Replicator Dynamics with Spatial Structure for Evolutionary Games

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Table of Contents

Evolutionary Games

Evolutionary Game Theory

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The Logic of Animal Conflict

J. MAYNARD SMITH
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G. R. PRICE
Galton Laboratory, University College London, 4 Stephenson Way, London NW1 2HE

It originated from the work of John Maynard Smith and George R. Price in 1973

Replicator Dynamics

The most famous way to describe an evolutionary game is through the replicator equation. If you have a well mixed infinitely large population and each player plays a single pure strategy then the proportion of players using the ith strategy, p_i changes in time by

$$\frac{d}{dt}p_i(t) = p_i(f_i(p) - \varphi(p))$$

where $p = [p_i]_{i=1}^m$, f_i is the fitness of playing strategy i against the mixture p and $\varphi(p)$ is the fitness of an average individual in the mixture $p \in \Delta^{m-1}$.

Replicator Dynamics

In the pure strategy coordination game your fitness is exactly the proportion of neighbors using the same strategy as you. Thus $f_i(p) = p_i$ and $\varphi(p) = |p|^2$

Coordination Example

For a pure coordination game. The replicator equation is

$$\frac{d}{dt}p_i = p_i(p_i - |p|^2)$$

All equilibria have the property that for all i such that $p_i \neq 0$ it is true that $p_i = |p|^2$. It is unstable when there is more than one such p_i.

Shifting Key Assumptions

Consider the setting where each player takes on a *mixed strategy*, there are a finite number of players where are not well mixed.

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Consider the setting where each player takes on a *mixed strategy*, there are a finite number of players where are not well mixed.

$$\frac{d}{dt}u_v^i = u_v^i(f_v^i(u_v) - \varphi_v(u_v))$$

Is this model still meaningful?

Notation

Consider the game with payoff matrix $A \in \mathbb{R}^{m \times m}$. If player 1 plays strategy $u_1 \in \Delta^{m-1}$ and player 2 plays strategy $u_2 \in \Delta^{m-1}$ then the payoff for player 1 is given as

$$w_1(u_1 | u_2) = \langle u_1, Au_2 \rangle$$

If the game is played on a graph with adjacency matrix W where each vertex $v \in V$ is a player, we say the game is *additive* or *linear* if

$$w_v(u_v | u) = \sum_{w \in V} W_{v,w} \langle u_v, Au_w \rangle$$

As a shorthand for this sum we write $g_v = \sum_{w \in V} W_{v,w} Au_w$

Structured Replicator Dynamics

Throughout most of this talk, high indecies indicate strategies, low indicies indicate players. With this convention for an additive game on the graph G adjacency matrix W and payoff matrix A, our model is written as

$$\frac{d}{dt} u_v^i = f_v^i(u) = u_v^i \langle \mathbf{e}^i - u_v, g_v \rangle$$

where $\mathbf{e}^i = [\delta_{i,j}]_{j=1}^m$.

Structured Replicator Dynamics

The model

$$\frac{d}{dt}u = f(u) = [u_v^i \langle \mathbf{e}^i - u_v, g_v \rangle]_{v \in V}^{i \in C} \tag{1}$$

The goal is to show that this model is well posed, makes sense as an evolutionary game theoretic model, and is useful for our understanding of structured games. I will show:

1. The model is well posed
2. The model satisfies the “folk theorem of EGT”
3. The model can improve our understanding of structured coordination

Model properties

Unique solutions to the IVP exist for a short time because $f \in C^\infty(\mathbb{R}^{m \cdot n})$. Moreover, solutions remain in the "biologically reasonable" domain and the model represents a "better reply" strategy revision protocol.

Theorem 1: $(\Delta^{m-1})^V$ is an invariant manifold for (1)

If u solves the initial value problem (1) with $u(0) \in (\Delta^{m-1})^V$ Then $u(t) \in (\Delta^{m-1})^V$ for all $t \geq 0$.

Theorem 2: The model (1) demonstrates a better reply dynamic.

If u is a solution to the initial value problem (1), all players are changing their strategy to increase their fitness relative to the present strategy profile. That is: $\langle \frac{\partial}{\partial t} u_v, \nabla_v w_v(u_v|u) \rangle \geq 0$

Potential Games

In classical game theory, a multiplayer game is a *potential game* if there exists a function U such that

$$U(u_i, u_{-i}) - U(u_i', u_{-i}) = w_i(u_i|u_{-i}) - w_i(u_i'|u_{-i})$$

If A is a symmetric matrix (more restrictive than being a symmetric game) then the additive game played on the graph G is a potential game with potential given as

$$\mathcal{W}(u) = \frac{1}{2}\sum_{v}\langle u_v, g_v\rangle = \frac{1}{2}\sum_{v\in V}\sum_{w\in V} W_{v,w}\langle u_v, u_w, \rangle \qquad (2)$$

Potential Games in the Evolutionary Setting

This is important in the evolutionary setting because it gives us the basis for a Lyapunov function

Theorem 3: Trajectories ascend the total fitness landscape

If $u(t)$ solves the IVP (1) with $(u(0) \in \Delta^{m-1})^V$ and A is a symmetric matrix, then

$$\frac{d}{dt}\mathcal{W}(u(t)) \geq 0$$

with equality only achieved if $\frac{d}{dt}u = 0$

This means we can define basins of stability for equilibria to (1) by examining superlevel sets of $\mathcal{W}$.

Converging to Equilibria

Lastly, it is important to note that if a trajectory converges to a point $u^\star \in (\Delta^{m-1})^V$ then $u^\star$ is a Nash equilibrium.

Nash Equilibrium

A strategy profile $u^\star$ is a Nash equilibrium if and only if $u_v \in br_v(u^\star)$ for all v

here $br_v(u^\star)$ is the mixed strategy best response of player v to the strategy profile $u^\star$. That is, the set of all mixed strategies s, such that $w_v(s|u^\star) = \max_{x\in\Delta^{m-1}} w_v(x|u^\star)$. Because w_v is linear with respect to player v's strategy, we can say

$$u_v \in br_v(u^\star) \iff C(u_v) \subset BR_v(u^\star)$$

where $C(u_v)$ is the support of u_v and $BR_v(u^\star)$ is the set of pure strategy best responses.

Converging to Equilibria

With this definition I can state the following result:

Theorem 4: Convergent trajectories converge to Nash equilibria

If $u(t)$ solves the IVP (1) is a convergent trajectory in the interior of $(\Delta^{m-1})^V$, then $\lim_{t\to\infty} u(t)$ is a Nash equilibrium.

proof: Suppose that $lim_{t\to\infty}u(t) = u^\star$ was not a Nash equilibrium so for some v and i, $g_v^{\star i} - \langle u_v^\star, g_v^\star \rangle > \delta > 0$. This quantity is continuous in u so surely there is a ν so that $|u - u^\star| < \nu \implies g_v^i - \langle u_v, g_v \rangle > \frac{\delta}{2} > 0$. By supposition there is a T so that

$$t > T \implies |u(t) - u^\star| < \nu \implies g_v^i(t) - \langle u_v(t), g_v(t) \rangle > \frac{\delta}{2}$$

Thus, or $t > T$, $\frac{d}{dt}u_v^i(t) \geq \frac{\delta}{2}u_v^i$ which is a contradiction.

Folk Theorem of Evolutionary Game Theory

In Evolutionary Game Theory, a dynamic game theoretic model is considered against the Folk Theorem of Evolutionary Game Theory to determine if it is useful as an evolutionary model.

Folk theorem of Evolutionary Game Theory (Cressman & Apaloo)

a. A stable rest point is a Nash Equilibrium.
b. A convergent trajectory on the interior of $(\Delta^{m-1})^V$ evolves to a Nash Equilibrium.
c. A strict Nash equilibrium is locally asymptotically stable

Folk Theorem of Evolutionary Game Theory

Theorem 4 is exactly part (b) of the folk theorem. To prove part (c) we will first compute the Jacobian for the system (1) but we will need other methods to prove part (a)

Folk theorem of Evolutionary Game Theory (Cressman & Apaloo)

a. A stable rest point is a Nash Equilibrium.

✓. A convergent trajectory on the interior of $(\Delta^{m-1})^V$ evolves to a Nash Equilibrium.

c. A strict Nash equilibrium is locally asymptotically stable

Computing the Jacobian

We easily compute partials

$$\begin{aligned}
\frac{\partial}{\partial u_v^i} f_v^i(u) &= \langle \mathbf{e}^i - u_v, g_v \rangle - u_v^i g_v^i \\
\frac{\partial}{\partial u_v^j} f_v^i(u) &= -u_v^i g_v^j \\
\frac{\partial}{\partial u_w^i} f_v^i(u) &= u_v^i \langle \mathbf{e}^i, A(\mathbf{e}^i - u_v \rangle) W_{w,v} \\
\frac{\partial}{\partial u_w^j} f_v^i(u) &= u_v^i \langle \mathbf{e}^j, A(\mathbf{e}^i - u_v \rangle) W_{w,v}
\end{aligned} \tag{3}$$

Then we can imagine the Jacobian as a block matrix with each block $J_{v,w} = \left[\frac{\partial}{\partial u_w^j} f_v^i\right]_{i,j \in C^2} \in \mathbb{R}^{m \times m}$

Analyzing the Jacobian

When we notice that for pure strategies $\frac{\partial}{\partial u_w^j} f_v^i = 0$ for $w \neq v$ the analysis of the Jacobian becomes very simple:

lemma 2: Spectrum of $J(u^\star))$ for pure strategies

If $u^\star$ is a pure strategy equilibrium then the spectrum of the Jacobian of the ODE system (1) evaluated at $u^\star$ is

$$\sigma(J(u^\star)) = \bigcup_{v \in V} \left(\{-\langle u_v, g_v \rangle\} \cup \{(g_v^i - \langle u_v, g_v \rangle) : i \notin C(u_v)\} \right)$$

Moreover we can easily define the eigenspaces for each eigenvalue and find that all eigenvalues are non-defective.

Folk Theorem of Evolutionary Game Theory

From this we can prove part (c) of the folk theorem of EGT rather simply.

Theorem 5: Folk Theorem of EGT (c)

If $u^\star$ is a strict Nash equilibrium then it is a locally asymptotically stable fixed point in the ODE system (1)

proof: If $u^\star$ is a strict Nash equilibrium then by the Equal payoff principle it is a pure strategy. Every eigenvalue in the form of a relative payoff must be strictly negative.

$$\sigma(J(u^\star)) = \bigcup_{v \in V} (\{-\langle u_v, g_v \rangle\} \cup \{(g_v^i - \langle u_v, g_v \rangle) : i \notin C(u_v)\})$$

As long as G is connected, it is no loss of generality to say that $-\langle u_v, g_v \rangle < 0$ for all v (Lemma 1).

Folk Theorem of Evolutionary Game Theory

Folk theorem of Evolutionary Game Theory (Cressman & Apaloo)

a. A stable rest point is a Nash Equilibrium.

✓. A convergent trajectory on the interior of $(\Delta^{m-1})^V$ evolves to a Nash Equilibrium.

✓. A strict Nash equilibrium is locally asymptotically stable

This is great in the case of strict Nash equilibria but even for pure strategy Nash equilibria which are not strict we run into issues with $J(u^\star)$ having a nontrivial kernel meaning that we cannot use Hartman-Grobman to determine stability. We will take a different approach with part (a)

Folk Theorem of Evolutionary Game Theory

Theorem 6: Folk Theorem of Evolutionary Game Theory (a)

If $u^\star$ is a rest point of the ODE system 1 and $u^\star$ is *not* a Nash equilibrium then $u^\star$ is unstable

Proof: $u^\star$ is not a Nash equilibrium so there is some player strategy pair v, i for which $g_v^\star i - \langle u_v^\star, g_v^\star \rangle > 0$ and $u_v^{\star i} = 0$ Let $x(t) = u(t) - u^\star$ and consider

$$\frac{d}{dt} x_v^i = (u_v^{\star i} + x_v^i) \langle \mathbf{e}^i - u_v^\star - x_v, g_v^\star + \sum_{w \in V} W_{v,w} A x_w \rangle$$

Moreover $u_v^{\star i} = 0$ so we can further reduce this ODE.

Folk Theorem of Evolutionary Game Theory

Proof continued: Expand the inner product and describe each x_v as $|x_v|\eta_v(t)$ where $\eta_v \in S^{m-1}$.

$$\frac{d}{dt}x_v^i = x_v^i(g_v^{\star i} - \langle u_v^\star, g_v^\star\rangle - |x_v|\langle \eta_v, g_v^\star\rangle + \sum_{w\in V} |x_w| W_{v,w}(\langle \mathbf{e}^i, A\eta_w\rangle - \langle u_v, A\eta_w\rangle - |x_v|\langle \eta_v, A\eta_w\rangle))$$

Note that because $\eta \in S^{m-1}$ and every term is continuous we can write that $|\langle \eta_v, g_v^\star\rangle| < c_v$, $|\langle \mathbf{e}^i, A\eta_w\rangle| \leq c_w$, $|\langle u_v, A\eta_w\rangle| < c_{v,w}^{(1)}$ and $|\langle x_v, Ax_w\rangle| < c_{v,w}^{(2)}$.

Folk Theorem of Evolutionary Game Theory

Proof continued: So we can control the impact of our higher order terms easily

$$|B(x)| := \Big| -|x_v|\langle \eta_v, g_v^\star \rangle + \sum_{w \in V} |x_w| W_{v,w}(\langle \mathbf{e}^i, A\eta_w \rangle - \langle u_v^\star, A\eta_w \rangle - |x_v|\langle \eta_v, A\eta_w \rangle) \Big| < |x_v|c_v + \sum_{w \in V} |x_w|(c_w + c_{v,w}^{(1)} + |x_v|c_{v,w}^{(2)}) < |x|C \quad (4)$$

This gives us the helpful differential inequality

$$\frac{d}{dt}x_v^i \geq x_v^i(g_v^{\star i} - \langle u_v^\star, g_v^\star \rangle - C|x|)$$

Folk Theorem of Evolutionary Game Theory

Proof continued: We know that $g_v^{\star i} - \langle u_v^\star, g_v^\star \rangle > \delta > 0$ so whenever $|x| < \frac{\delta}{2C}$ we have that

$$\frac{d}{dt} x_v^i \geq x_v^i (g_v^{\star i} - \langle u_v^\star, g_v^\star \rangle - C|x|) \geq \frac{\delta}{2} x_v^i$$

Therefore, any perturbation x with $x_v^i > 0$ will eventually leave the neighborhood $B_{\frac{\delta}{2C}}(u^\star)$ and thus $u^\star$ is unstable.

Folk Theorem of Evolutionary Game Theory

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Folk theorem of Evolutionary Game Theory (Cressman & Apaloo)

- ✓ A stable rest point is a Nash Equilibrium.
- ✓. A convergent trajectory on the interior of $(\Delta^{m-1})^V$ evolves to a Nash Equilibrium.
- ✓. A strict Nash equilibrium is locally asymptotically stable

The Pure Coordination Game

Having shown that this model is meaningful in an evolutionary game theoretic context, we want to use it to understand the Pure Coordination game better ($A = I_m$).

- I_m is obviously symmetric so we have a potential function $\mathcal{W}$. This potential function may help us explain why certain behaviors are not present in the classical game
- We now have a mapping

$$\Phi : \mathbb{R}^{n \times n} \to C^{\infty}(\mathbb{R}^{n \cdot m}; \mathbb{R})$$

  which maps $W \mapsto \mathcal{W}$ as a way for understanding the relationship between structure and coordination behavior
- We can classify further the behavior of solutions in the vicinity of Nash equilbria.

The Pure Coordination Game

Lemma 5: Spectrum of the Mixed strategy Jacobian

If $u^\star$ is a mixed strategy profile in the pure coordination game, and if one of the following is true for every pair v, w:

1. $|C(u_v^\star)| = 1$ or $|C(u_w^\star)| = 1$
2. $W_{v,w} = 0$
3. $C(u_v^\star) \cap C(u_w^\star) = \emptyset$

Then the spectrum of the Jacobian evaluated at $u^\star$ is given as

$$\sigma(J(u^\star)) = \bigcup_{v \in V} \left(\{-\langle u_v^\star, g_v^\star \rangle, 0\} \bigcup_{i \notin C(u_v^\star)} \{g_v^{\star i} - \langle u_v^\star, g_v^\star \rangle\} \right)$$

where 0 has algebraic multiplicity $|BR_v(u^\star)| - 1$ for each v.

The Pure Coordination Game

Lemma 6: Overlapping best responses are unstable

For an Nash equilibrium $u^\star$, in a game with a symmetric payoff matrix A, If at least two adjacent players v and w satisfy the following

h1) There exists an $i \in BR_v(u^\star) \cap BR_w(u^\star)$ such one of the following holds:
(a) $u_v^{\star i} > 0$ and $u_w^{\star i} > 0$
(b) $u_v^{\star i} < 1$ and $u_w^{\star i} < 1$

h2) $|BR_v(u^\star)| > 1$, and $|BR_w(u^\star)| > 1$

then $u^\star$ is unstable.

Proof of Lemma 6

Proof: We will use a Chetaev instability argument with the Chetaev function $\mathcal{W}(u^\star + x) - \mathcal{W}(u^\star)$. Let

$$U_{u^\star} = \{x \in \mathbb{R}^{m \cdot n}; \mathcal{W}(u^\star + x) - \mathcal{W}(u^\star) > 0\}$$

Then $U_{u^\star}$ is open and $0 \in \partial U_{u^\star}$.
We also note that if $x \in U_{u^\star}$ then

$$\sum_{v \in V} \langle x_v, g_v^\star \rangle + \frac{1}{2} \sum_{v \in V} \sum_{w \in V} W_{v,w} \langle x_v, x_w \rangle > 0 \tag{5}$$

and thus for some v, $\langle x_v, g_v^\star \rangle + \frac{1}{2} \sum_{w \in V} W_{v,w} \langle x_v, x_w \rangle > 0$.

Proof of Lemma 6

Now we examine $\frac{d}{dt}u$ in this region. For the v selected above, select an i for which $u_v^{\star i} + x_v^i > 0$. Such an $i \in C(u_v^\star)$ always exists when $|x| < \epsilon$. Now we can compute

$$\frac{d}{dt}u_v^i = \underbrace{(u_v^{\star i} + x_v^i)}_{>0}\left(\underbrace{-\langle x_v, g_v^\star\rangle - \sum_{w\in V} W_{v,w}\langle x_v, x_w\rangle}_{<-\delta} + \sum_{w\in V} W_{v,w}\langle \hat{e}^i - u_v^\star, x_w\rangle\right)$$

so all that remains is to show this final sum is nonpositive for some $i \in C(u_v^\star)$. This is certainly true if we select $i \in C(u_v^\star)$ for which $\sum W_{v,w}x_w^i$ is minimized.

Proof of Lemma 6

Therefore, we can show that when (h1) and (h2) are satisfied, for any $x \in U_{u^\star} \cap B_\epsilon(0)$, $f(u^\star + x) \neq 0$. By theorem 3 this means that $\frac{d}{dt}\mathcal{W}(u^\star + x(t)) - \mathcal{W}(u^\star) > 0$ for all $x \in U_{u^\star} \cap B_\epsilon(0)$

Chetaev Instability Theorem (Chetaev 1962)

If $\dot{x} = f(x)$ is an ODE system with $f(0) = 0$ and there exists a continuously differentiable function V such that

1. $0 \in \partial U$ where $U := \{x \in \mathbb{R}^n; V(x) > 0\}$
2. $\frac{d}{dt}V(x(t)) > 0$ for all $x \in U \cap B_\epsilon(0)$ for some ϵ

Then $x = 0$ is unstable if $U \cap B_\epsilon(0)$ is non-empty.

The Pure Coordination Game

With these two results we can attempt to classify equilibria.

Classification of equilibria

Let $u^\star$ be a strategy profile for the pure coordination game played on the graph G with adjacency matrix W.

1. If $u^\star$ is not a Nash equilibrium then $u^\star$ is either not an equilibrium or an unstable equilibrium
2. If $u^\star$ is a Nash equilibrium that satisfies both (h1) and (h2) then it is an unstable equilibrium
3. If $u^\star$ is a Nash equilibrium which does not satisfy either (h1) or (h2) then it is Lyapunov Stable
4. If $u^\star$ is a *strict* Nash equilibrium then it is locally Asymptotically stable

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Let $u^\star$ be a strategy profile for the pure coordination game played on the graph G with adjacency matrix W.

- ✓ If $u^\star$ is not a Nash equilibrium then $u^\star$ is either not an equilibrium or an unstable equilibrium
- ✓ If $u^\star$ is a Nash equilibrium that satisfies both (h1) and (h2) then it is an unstable equilibrium
- 3. If $u^\star$ is a Nash equilibrium which does not satisfy either (h1) or (h2) then it is Lyapunov Stable
- ✓ If $u^\star$ is a *strict* Nash equilibrium then it is locally Asymptotically stable

Center manifold for case (3)

We cannot, at present, complete case (3) entirely but we can complete an interesting subcase.

Lemma 7: Characterization of the Center Manifold in case (3)

Suppose that $u^\star$ is a nonstrict Nash equilibrium to the pure coordination game. If, for any pair of players v, w one of the following holds:

1. $W_{v,w} = 0$
2. $|BR_v(u^\star)| = 1$ or $|BR_w(u^\star)| = 1$
3. $BR_v(u^\star) \cap BR_w(u^\star) = \emptyset$

Then $u^\star$ exists in a manifold of equilibria and solutions nearby $u^\star$ converge to this manifold.

The Pure Coordination Game

To complete the classification there are two remaining points:

- Show that in the subcase with the trivial center manifold, that convergence to the center manifold implies Lyapunov stability
- Consider the case where (h1) and (h2) are not satisfied but there are adjacent players with overlapping best responses

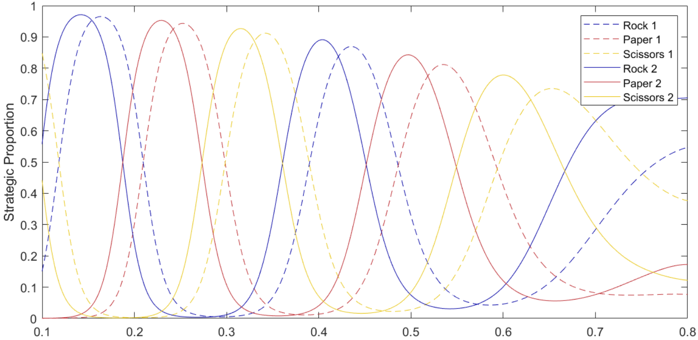
Future work on this system

There are many exciting future areas of study that this work permits:

- First complete the characterization of equilibria
- Use the potential function to investigate the absence of n-cycles in the discrete dynamic game for $n > 2$
- Examine $\Phi : W \mapsto \mathcal{W}$ to discuss how changing weights of edges in W changes the fitness landscape
- Consider continuous extensions of spatially explicit evolutionary games

Use as an evolutionary model in continuous space

If the population is large in a continuous patch but they mix only partially, these structured replicator dynamics can tell us something about the evolution of non-cooperative strategies propagating through space



Thank you

Questions?

Proof of Lemma 2

Having computed each of the partials we get that

$$J_{v,w} = \begin{cases} diag(g_v^i - \langle u_v, g_v \rangle) - u_v g_v^\intercal & w = v \\ 0 & w \neq v \end{cases}$$

So the Jacobian is block diagonal. Moreover, each diagonal block $J_{v,v}$ can be permuted into a upper triangular matrix by left and right multiplying the permutation matrix P_{1,k_v}. The diagonal elements of $J_{v,v}$ and $P_{1,k_v} J_{v,v} P_{1,k_v}$ are identical and the the eigenvalues are unchanged by the permutations so we get

$$\sigma(J_{v,v}(u^\star)) = \sigma(P_{1,k_v} J_{v,v}(u^\star) P_{1,k_v}) = \{-\langle u_v, g_v \rangle\} \cup \{g_v^i - \langle u_v, g_v \rangle\}_{i \neq k_v}$$

The final result is simply the union of the spectra for each $v \in V$.

Analyzing the Jacobian

All but n of the eigenvalues from lemma 2 are of the forms of relative payoff $g_v^i - \langle u_v, g_v \rangle = w_v(\mathbf{e}^i | u^\star) - w_v(u_v | u^\star)$.
The remaining n eigenvalues are of the form of $-\langle u_v, g_v \rangle = -w_v(u_v | u^\star)$. This seems to pose a problem because we have no control over absolute payoff.

Analyzing the Jacobian

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The remaining n eigenvalues are of the form of $-\langle u_v, g_v \rangle = -w_v(u_v | u^\star)$. This seems to pose a problem because we have no control over absolute payoff. luckily this problem can be solved in multiple ways. The easier way is the following lemma:

Lemma 1: The dynamics of (2) are invariant under constant addition to the payoff matrix

The dynamics of a solution to equation (1) with payoff matrix A are identical to the dynamics of a solution to the same equation with payoff matrix $A + B$ where B is some constant matrix.

Analyzing the Jacobian

This is nearly sufficient to prove part (c) of the folk theorem, but we need one more standard result from game theory

Equal payoff principle for mixed strategies (von Neumann and Morgenstern)

If a player is playing a mixed strategy in a Nash equilibrium, then the expected payoff of each pure strategy supported in their mixed strategy is equal

The result is that if $u^\star$ is a *strict* Nash equilibrium (meaning $|br_v(u^\star)| = 1$ for all v, then surely $u^\star$ is a pure strategy Nash equilibrium.

Center manifold for case (3)

Proof Idea: The center manifold theorem says that a center manifold exists and, because when $u^\star$ is a Nash equilibrium there is no unstable space, solutions in the neighborhood of $u^\star$ converge to a solution on the center manifold.

Center Manifold

For a ODE system $\frac{d}{dt}u = f(u)$ in the neighborhood of a non-hyperbolic equilibrium $x = 0$, we can separate the system into $x, y \in E_0 \times E_s$. There exists a center manifold W_c tangent to E_0 locally described by $h : E_0 \to E_s$. and for $(x, y) \in W_c$,

$$\frac{d}{dt}x = Ax + f(x, h(x))$$
$$\frac{d}{dt}y = Bh(x) + g(x, h(x))$$

Center manifold for case (3)

Properties of the center manifold

If h locally describes the center manifold then h must satisfy the differential equation

$$Dh(x)(Ax + f(x, h(x))) - Bh(x) - g(x, h(x)) = 0$$

This subcase is very easy because it happens that the center manifold, W^c is exactly the center space E_0. The height function $h : E_0 \to E^s$ which describes the center manifold is exactly $h(x) = 0$ for all $x \in E_0$.
Moreover, for $x \in E_0$ $f(u^\star + x) = 0$. Indeed the center manifold is a manifold of equilibria because out hypotheses have eliminated any nonlinear effects.