On Structured Models of Coordinating Systems

John McAlister
jmcalis6@vols.utk.edu

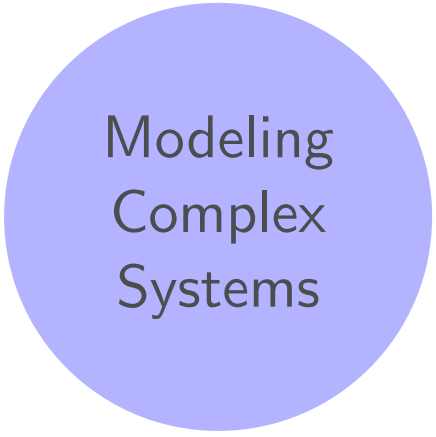
Invited Seminar in the School of Math and Natural Sciences,
Arizona State University, Phoenix Arizona

February 25th

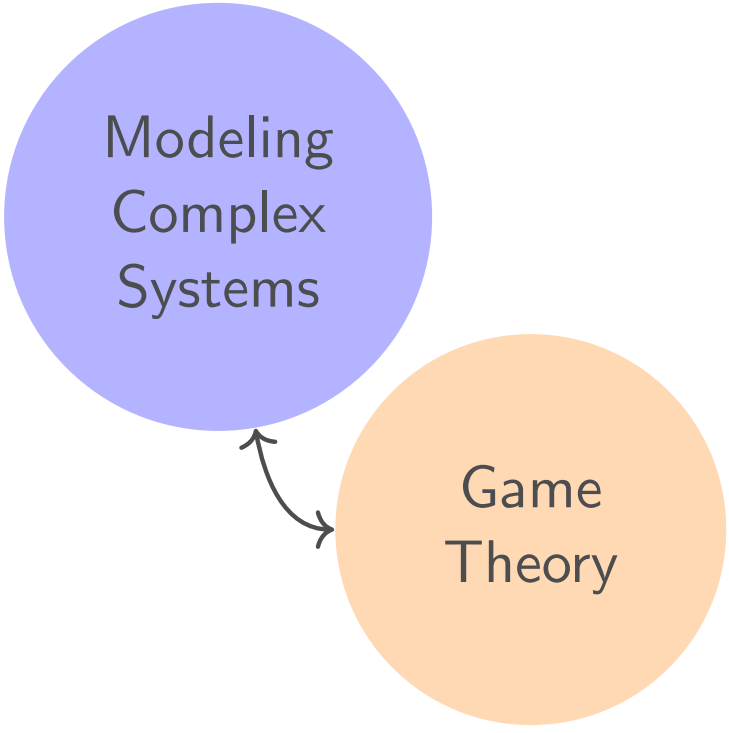
THE UNIVERSITY OF
TENNESSEE
KNOXVILLE

DEPARTMENT OF
MATHEMATICS

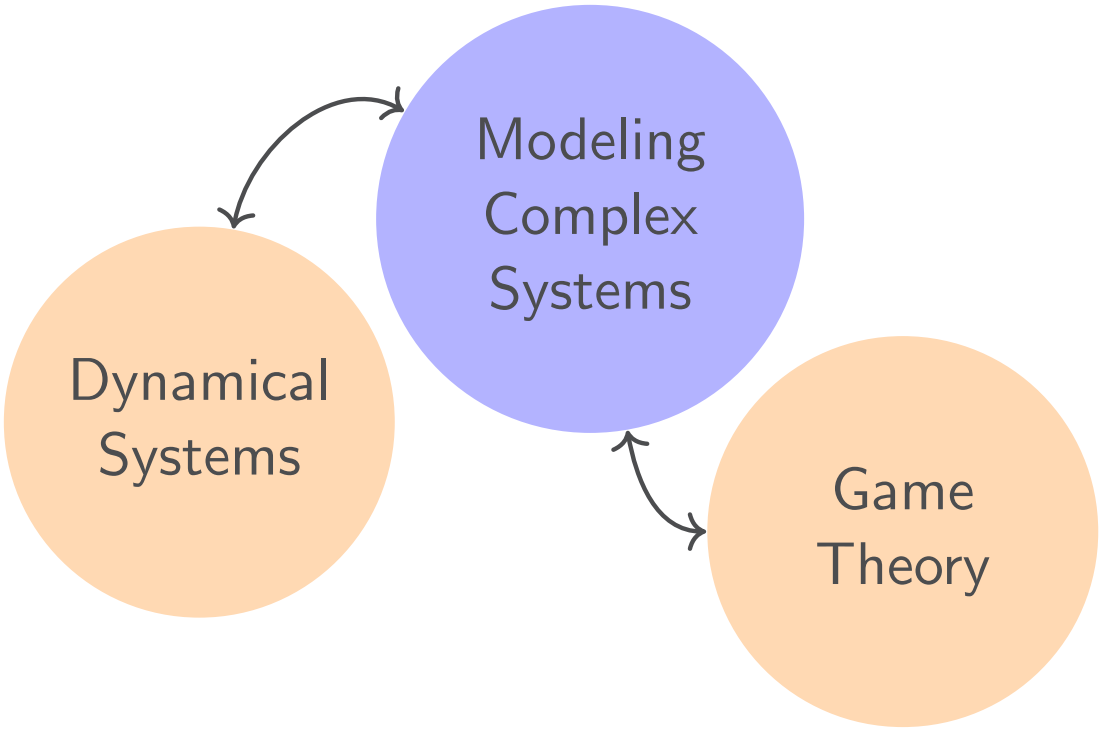
Overview



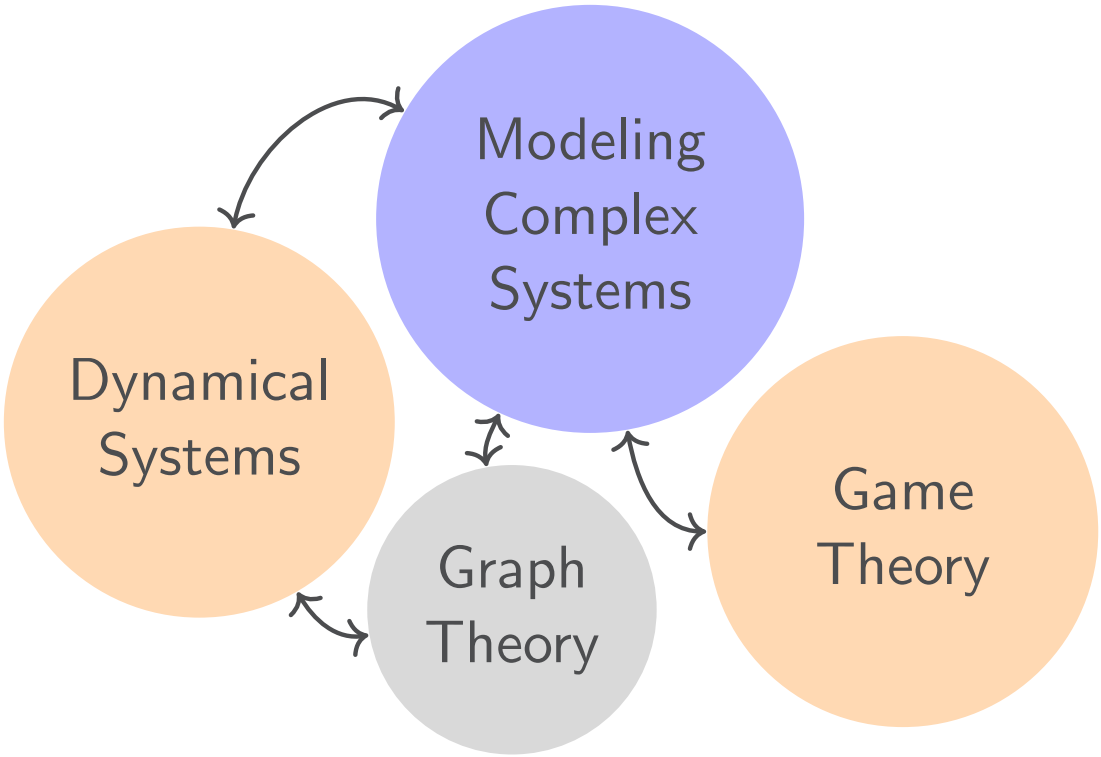
Overview



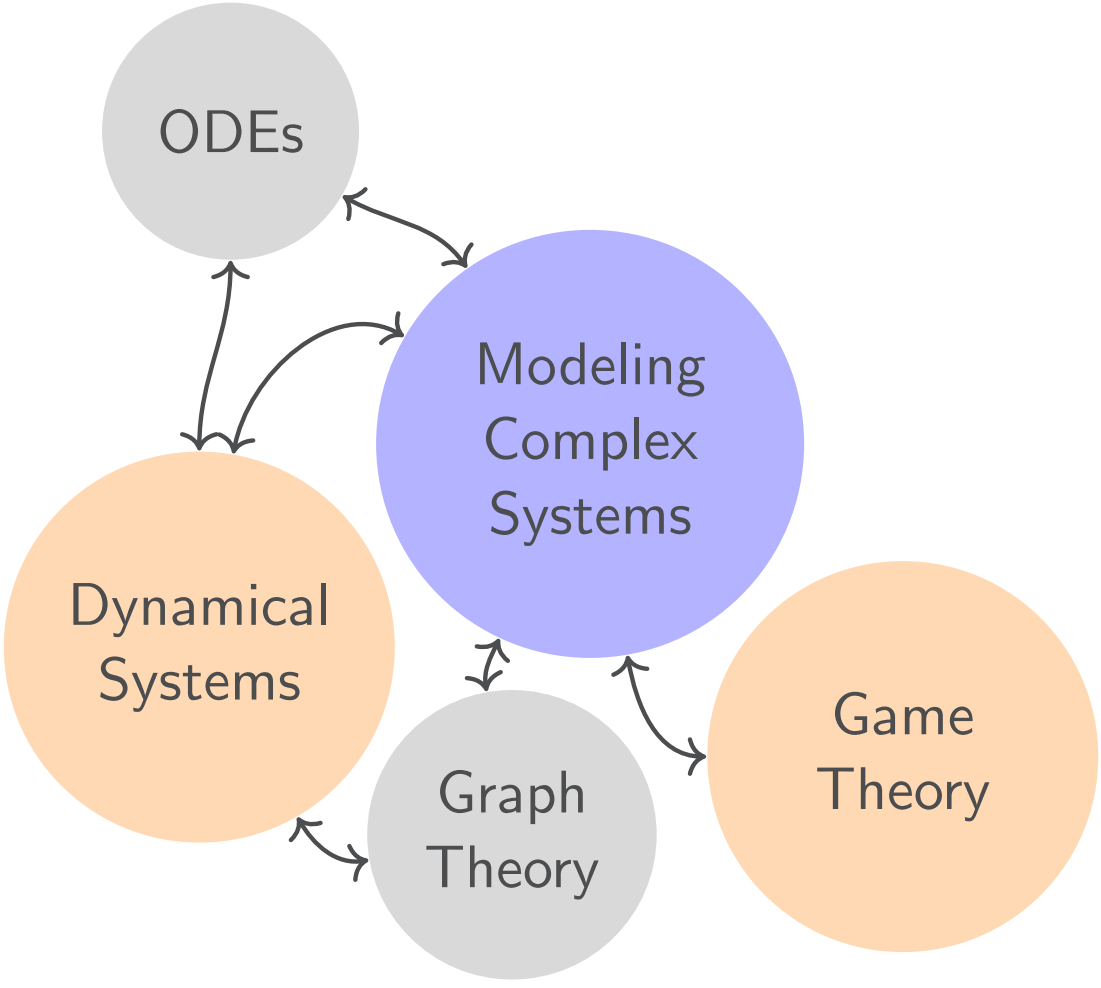
Overview



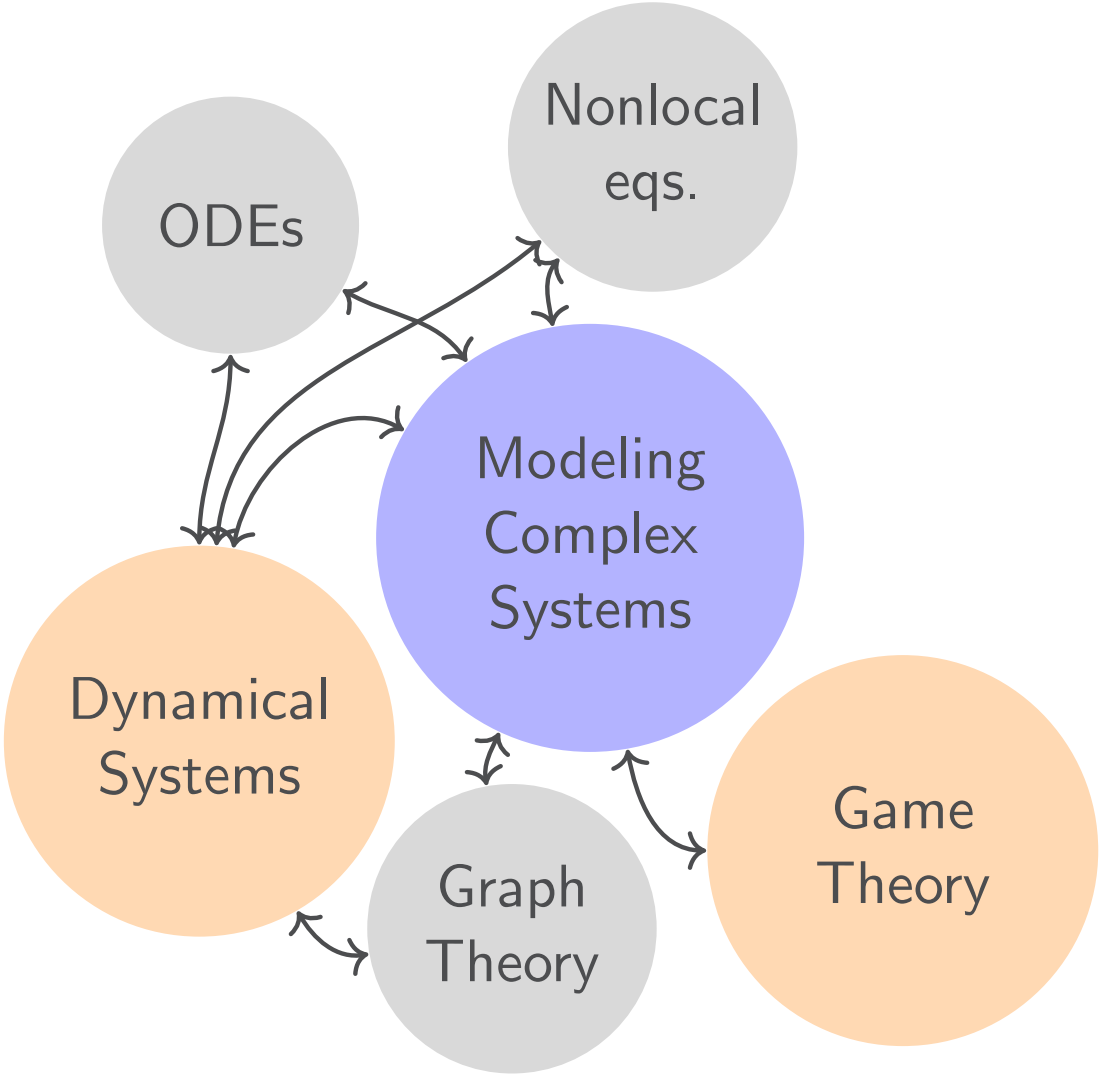
Overview


Modeling
Complex
Systems
Dynamical
Systems
Graph
Theory
Game
Theory

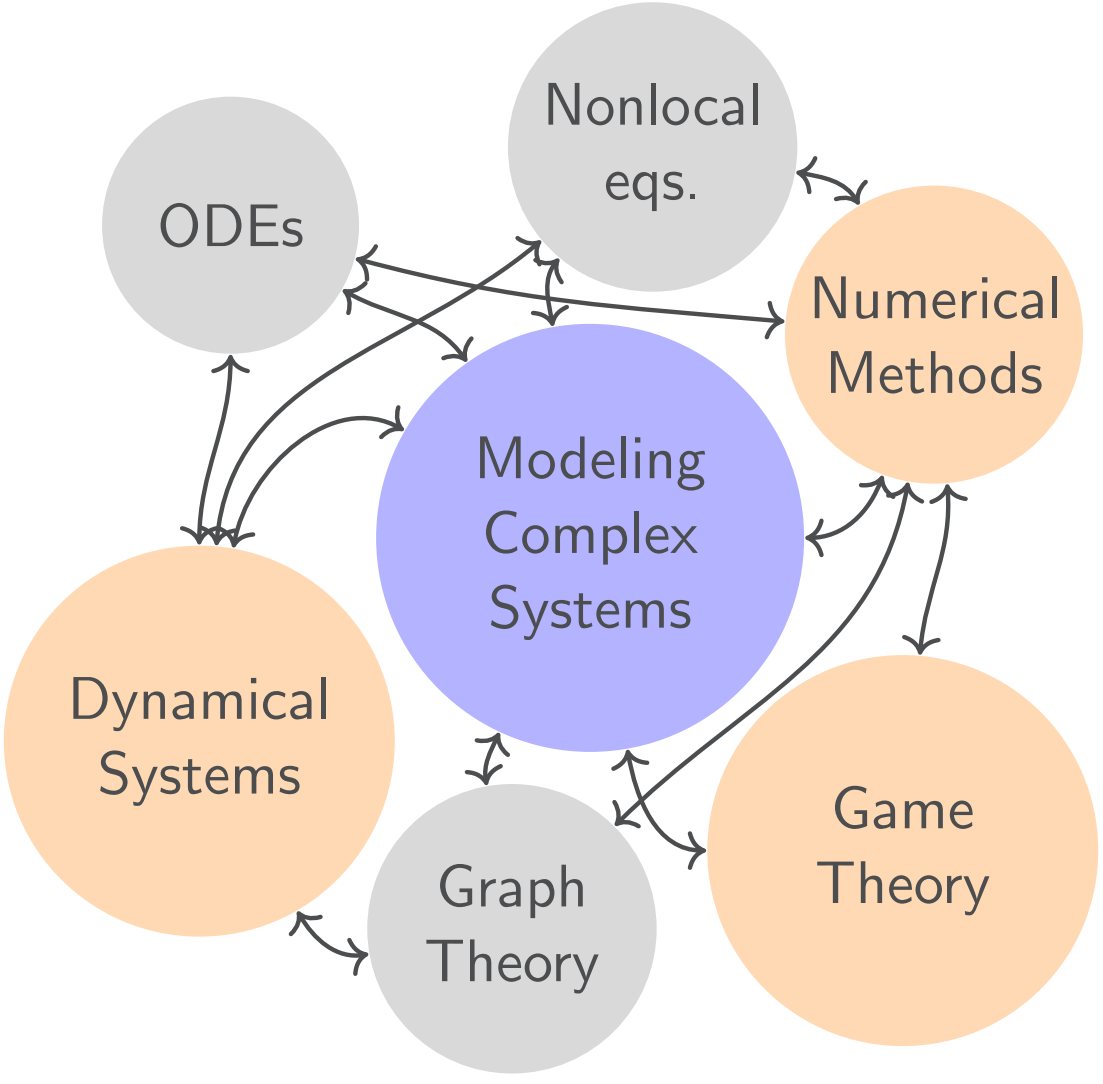
Overview


ODEs
Modeling
Complex
Systems
Dynamical
Systems
Graph
Theory
Game
Theory

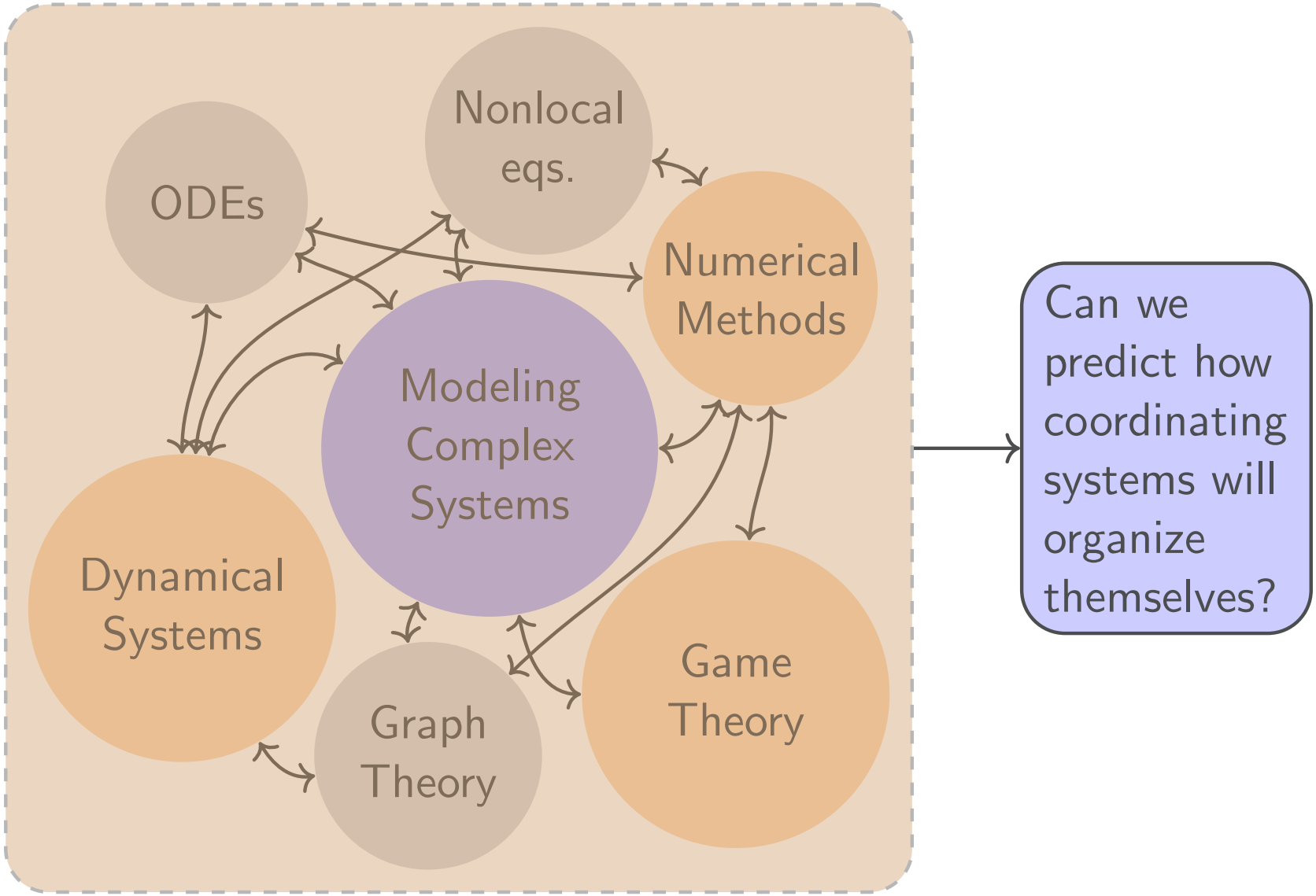
Overview


Nonlocal
eqs.
ODEs
Modeling
Complex
Systems
Dynamical
Systems
Graph
Theory
Game
Theory

Overview


Nonlocal
eqs.
ODEs
Numerical
Methods
Modeling
Complex
Systems
Dynamical
Systems
Game
Theory
Graph
Theory

Overview


Nonlocal eqs.
ODEs
Numerical Methods
Modeling Complex Systems
Dynamical Systems
Graph Theory
Game Theory
Can we predict how coordinating systems will organize themselves?

Overview

Introduction

Coordination

A fundamental game theoretic interaction in which two players receive a higher payoff by playing the same strategy than they would by playing different strategies.

Introduction

Coordination

A fundamental game theoretic interaction in which two players receive a higher payoff by playing the same strategy than they would by playing different strategies.

Introduction

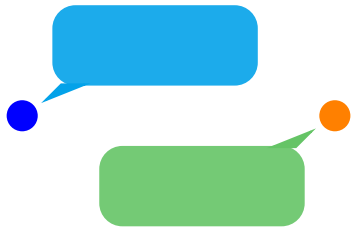
Coordination

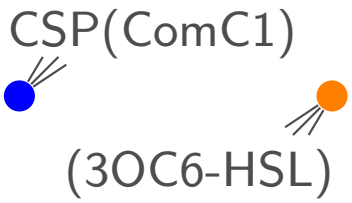
A fundamental game theoretic interaction in which two players receive a higher payoff by playing the same strategy than they would by playing different strategies.

Introduction

Coordination

A fundamental game theoretic interaction in which two players receive a higher payoff by playing the same strategy than they would by playing different strategies.





Definitions

To give a more rigorous description of the game we need the following definitions. For a game with players V and strategy set C:

- $u : V \to C$ is a strategy profile. describes which players play which strategies
- $w_v(c|u)$ is the Payoff for player v playing strategy c against the strategy profile u
- $BR_v(u) = \text{argmax}_{c \in C}\, w_v(c|u)$ is player v's best response to u
- u is a Nash equilibrium if $u_v \in BR_v(u)$ for all $v \in V$.

Bandwagon Property

Bandwagon Property (Kandori et al. 1993)

A game is a coordination game if and only if it satisfies the bandwagon property: for any strategy profile u,

$$BR_v(u) \subseteq C(u) \quad \forall v \in V$$

where $C(u)$ is the support of a strategy profile u (i.e. the set of all strategies being used)

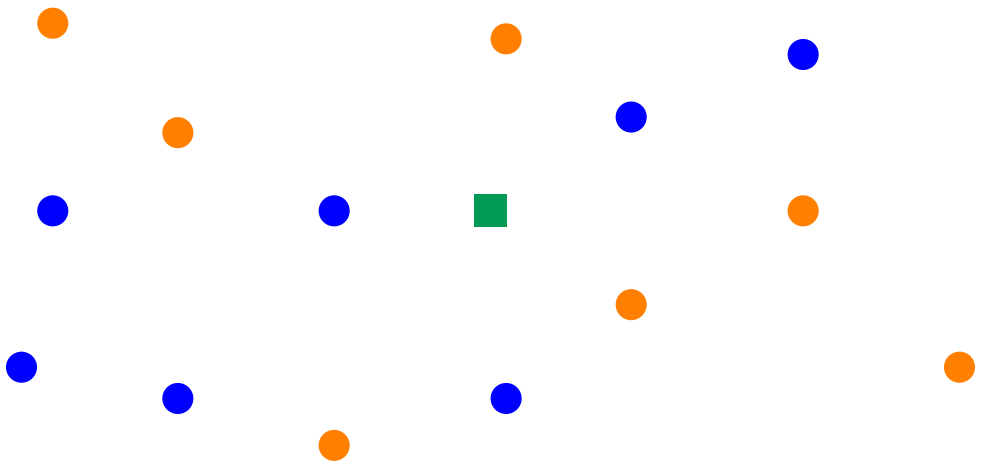
Bandwagon Property

Bandwagon Property (Kandori et al. 1993)

A game is a coordination game if and only if it satisfies the bandwagon property: for any strategy profile u,

$$BR_v(u) \subseteq C(u) \quad \forall v \in V$$

where $C(u)$ is the support of a strategy profile u (i.e. the set of all strategies being used)



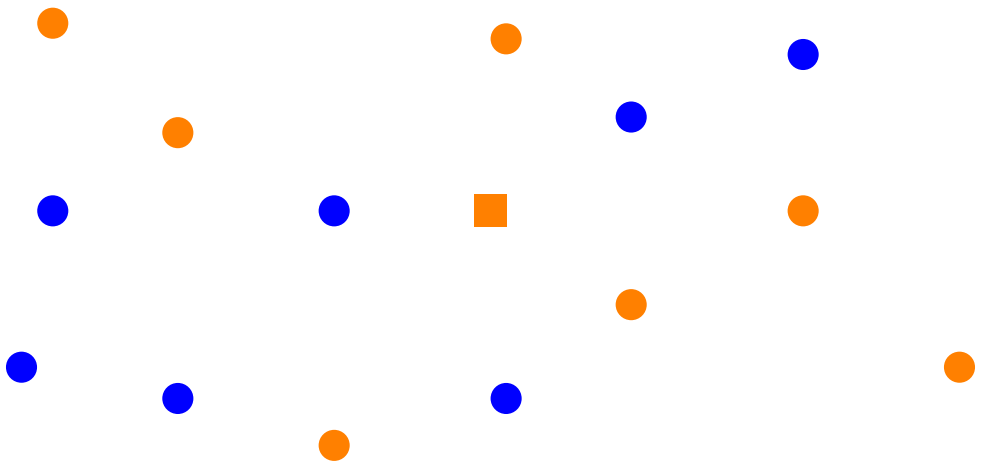
Bandwagon Property

Bandwagon Property (Kandori et al. 1993)

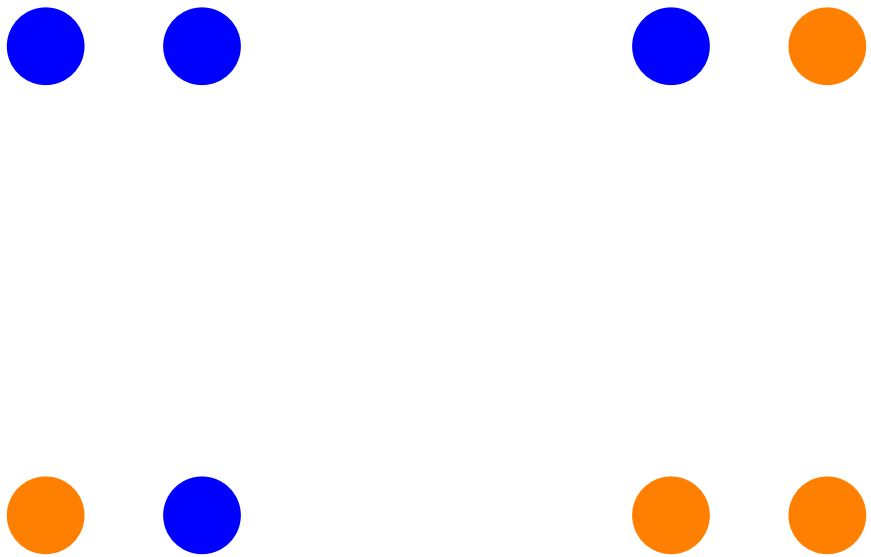
A game is a coordination game if and only if it satisfies the bandwagon property: for any strategy profile u,

$$BR_v(u) \subseteq C(u) \quad \forall v \in V$$

where $C(u)$ is the support of a strategy profile u (i.e. the set of all strategies being used)

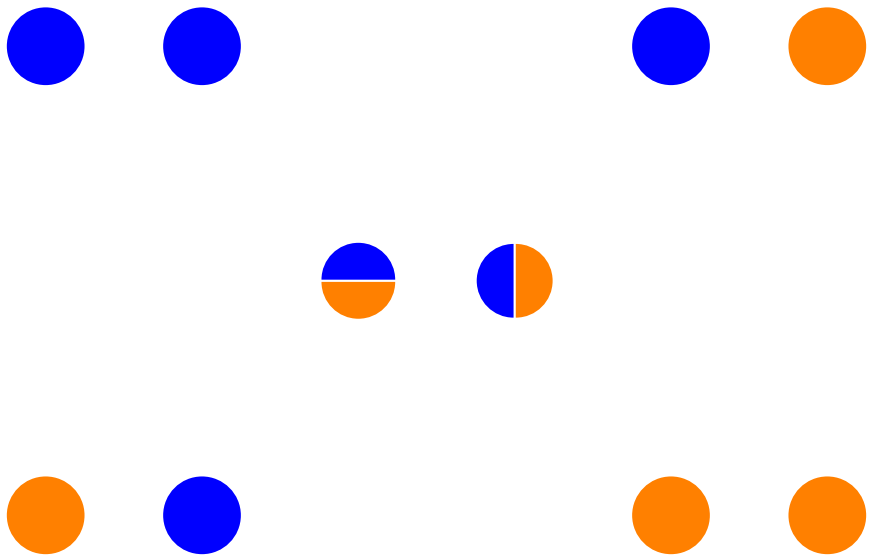


Two Player Coordination Game



The two player coordination game is entirely solved. There are three Nash equilibria. Two pure strategy Nash equilibria (consensus) and one mixed strategy Nash equilibrium (mixed consensus)

Two Player Coordination Game



The two player coordination game is entirely solved. There are three Nash equilibria. Two pure strategy Nash equilibria (consensus) and one mixed strategy Nash equilibrium (mixed consensus)

Two Player Coordination

The diagram encodes the payoff matrix for any coordination game

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where $a > c$ and $d > b$.
Using this to describe the two player coordination game more rigorously:

- $V = \{1, 2\}$
- $C = \{[1, 0], [0, 1]\}$
- $u = [u_1, u_2]$
- $w_v(u_v|u_w) = u_v^\intercal A u_w$ (and so if $A = I_2$ then $w_v(u_v|u_w) = \langle u_v, u_w \rangle$)

Two Player Coordination

The diagram encodes the payoff matrix for any coordination game

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

In the case of the pure coordination game.
Using this to describe the two player coordination game more rigorously:

- $V = \{1, 2\}$
- $C = \{[1, 0], [0, 1]\}$
- $u = [u_1, u_2]$
- $w_v(u_v|u_w) = u_v^T A u_w$ (and so if $A = I_2$ then $w_v(u_v|u_w) = \langle u_v, u_w \rangle$)

Dynamics of the Structured Coordination Game

To understand the systems dynamically, we will use the Myopic Best Response (MBR) strategy revision protocol.

Myopic Best Response

A sequence of strategy profiles $u(t)$ satisfies the myopic best response strategy revision protocol if, for all t,

$$u_v(t+1) \in BR_v(u(t)) \quad \forall v \in V$$

Dynamics of the Structured Coordination Game

To understand the systems dynamically, we will use the Myopic Best Response (MBR) strategy revision protocol.

Myopic Best Response

A sequence of strategy profiles $u(t)$ satisfies the myopic best response strategy revision protocol if, for all t,

$$u_v(t+1) \in BR_v(u(t)) \quad \forall v \in V$$

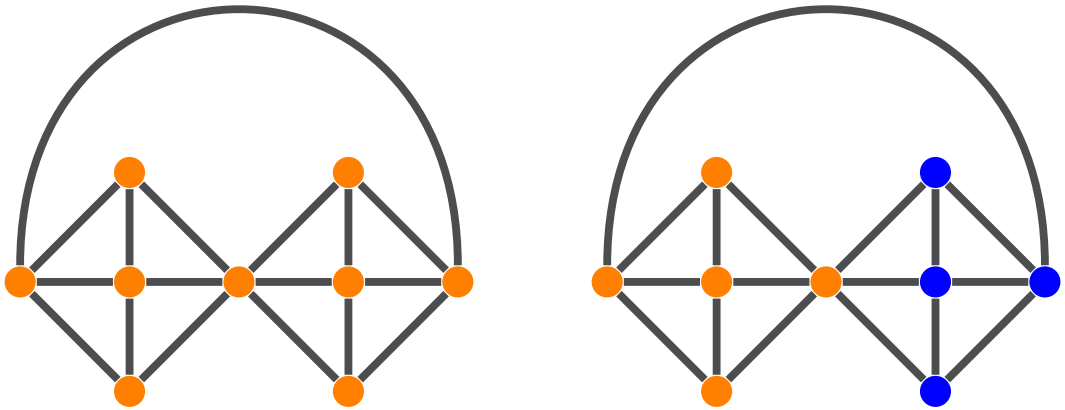
MBR sequences

If a MBR sequence converges to a limit, its limit is a Nash equilibrium.

The Structured Coordination Game

For the Structured Coordination Game played on the graph G with adjacency matrix W, the payoff function is

$$w_v(u_v|u) = \sum_{w \in V} W_{v,w} \langle u_v, u_w \rangle$$



Consensus is always a Nash equilibrium, but some graphs admit non-consensus Nash equilibria.

The Structured Coordination Game by Other Names

Central Question

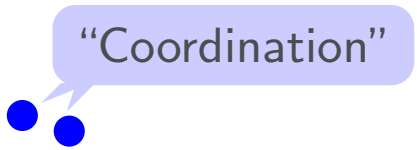
Given some graph, what non-consensus equilibria does it admit?

Many domains use a different name to refer to the same type of question. This itself is a coordination game.

The Structured Coordination Game by Other Names

Central Question

Given some graph, what non-consensus equilibria does it admit?

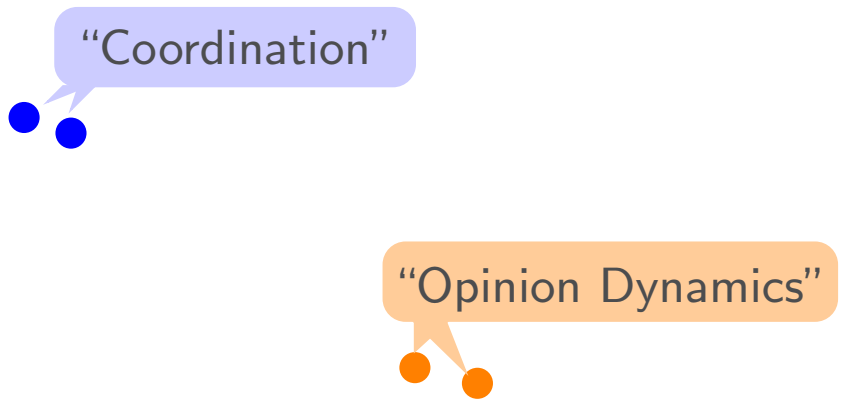


Many domains use a different name to refer to the same type of question. This itself is a coordination game.

The Structured Coordination Game by Other Names

Central Question

Given some graph, what non-consensus equilibria does it admit?

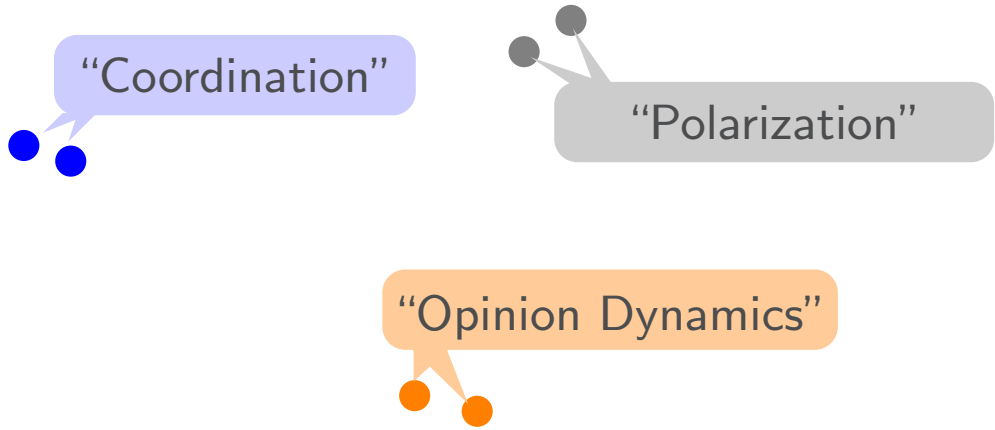


Many domains use a different name to refer to the same type of question. This itself is a coordination game.

The Structured Coordination Game by Other Names

Central Question

Given some graph, what non-consensus equilibria does it admit?

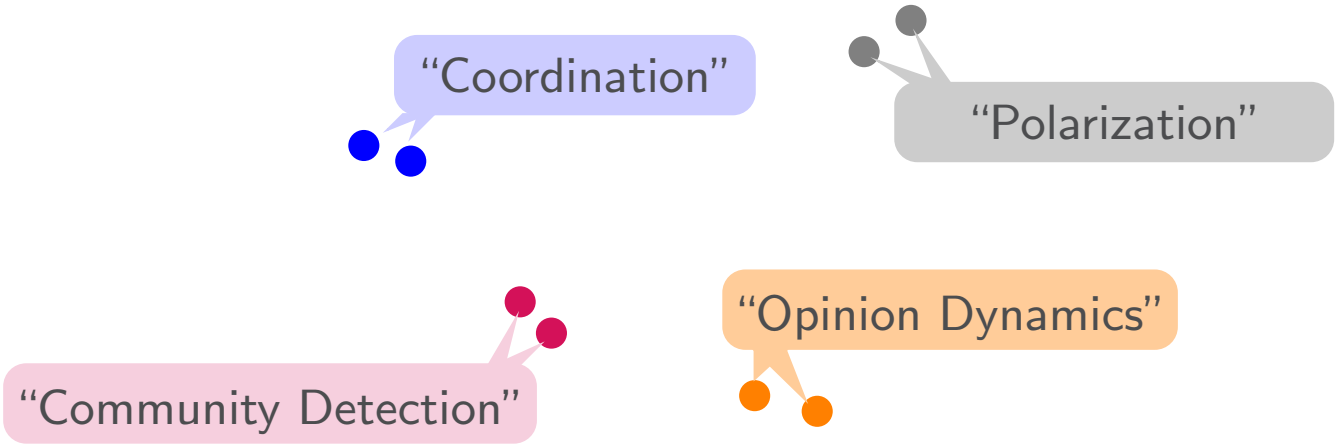


Many domains use a different name to refer to the same type of question. This itself is a coordination game.

The Structured Coordination Game by Other Names

Central Question

Given some graph, what non-consensus equilibria does it admit?

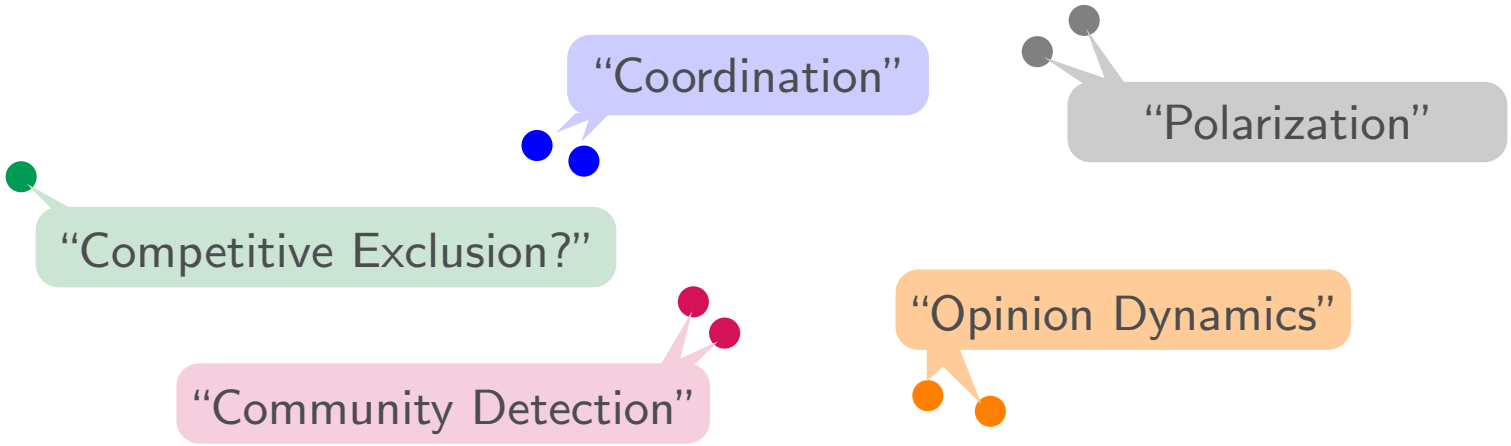


Many domains use a different name to refer to the same type of question. This itself is a coordination game.

The Structured Coordination Game by Other Names

Central Question

Given some graph, what non-consensus equilibria does it admit?



Many domains use a different name to refer to the same type of question. This itself is a coordination game.

Modeling Challenges

Every attempt to model a coordinating system must contend with these three challenges:

Numerical Speed	Can the model work quickly in large domains?
Analytical Tractability	Can we use the model to prove general results?
Model Accuracy	Does the model use best response dynamics? Does the model use the correct payoff function?

Modeling Challenges

Every attempt to model a coordinating system must contend with these three challenges:

Numerical Speed	Can the model work quickly in large domains?
Analytical Tractability	Can we use the model to prove general results?
Model Accuracy	Does the model use best response dynamics? Does the model use the correct payoff function?

Model Diversity

No model will be able to overcome all of these challenges, but through a wide range of models we can improve our understanding of coordinating systems.

Simulation Study

Our first modeling approach is through simulation

Dynamic Games and Applications
https://doi.org/10.1007/s13235-024-00612-4

Check for updates

Insights into the Structured Coordination Game with Neutral Options Through Simulation

John S. McAlister[1,3] · Nina H. Fefferman[1,2,3]

Accepted: 30 November 2024

Simulation set up

We focus on the setting where $A = I_m$ where m is the number of strategies available.

Payoff Function

$$w_v(u_v|u) = \sum_{w \in V} W_{v,w} \langle u_v, u_w \rangle$$

Strategy Revision Protocol

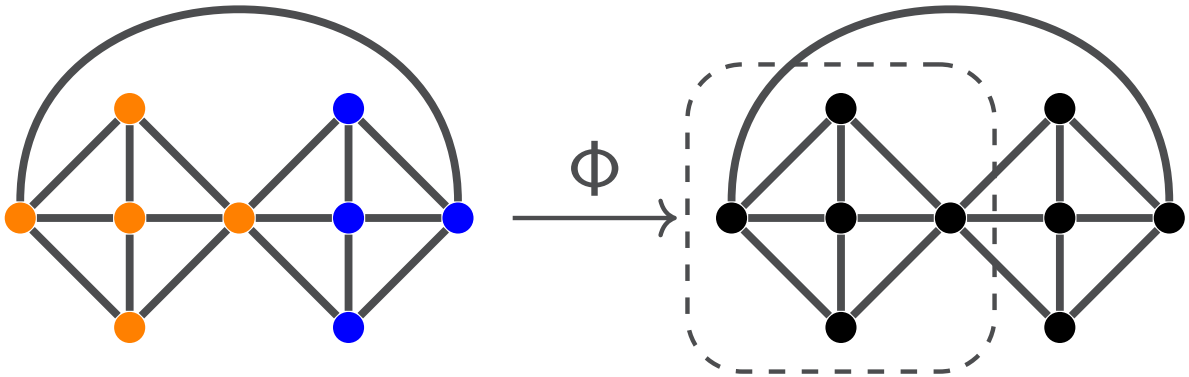
$$u_v(t+1) \in BR_v(u(t))$$

If $|BR_v(u(t))| > 1$ and $u(t) \in BR_v(u(t))$ then $u(t+1) = u(t)$ it $u(t) \notin BR_v(u(t))$ then $u(t+1)$ is drawn uniformly randomly.

Partitioning Approach

Strategy Space

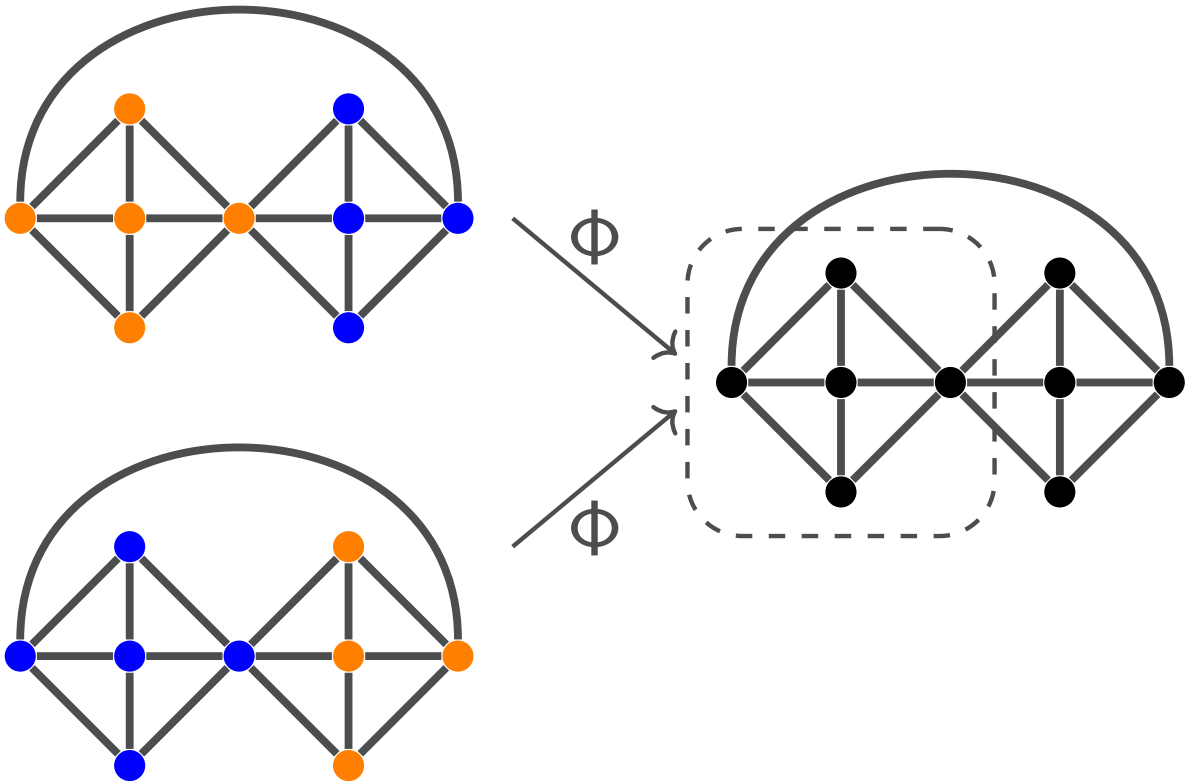
Let X_G be the set of all strategy profiles on G and let $\mathcal{Q}_G$ be the set of all vertex partitions of G. Define $\Phi : X_G \to \mathcal{Q}_G$



Partitioning Approach

Strategy Space

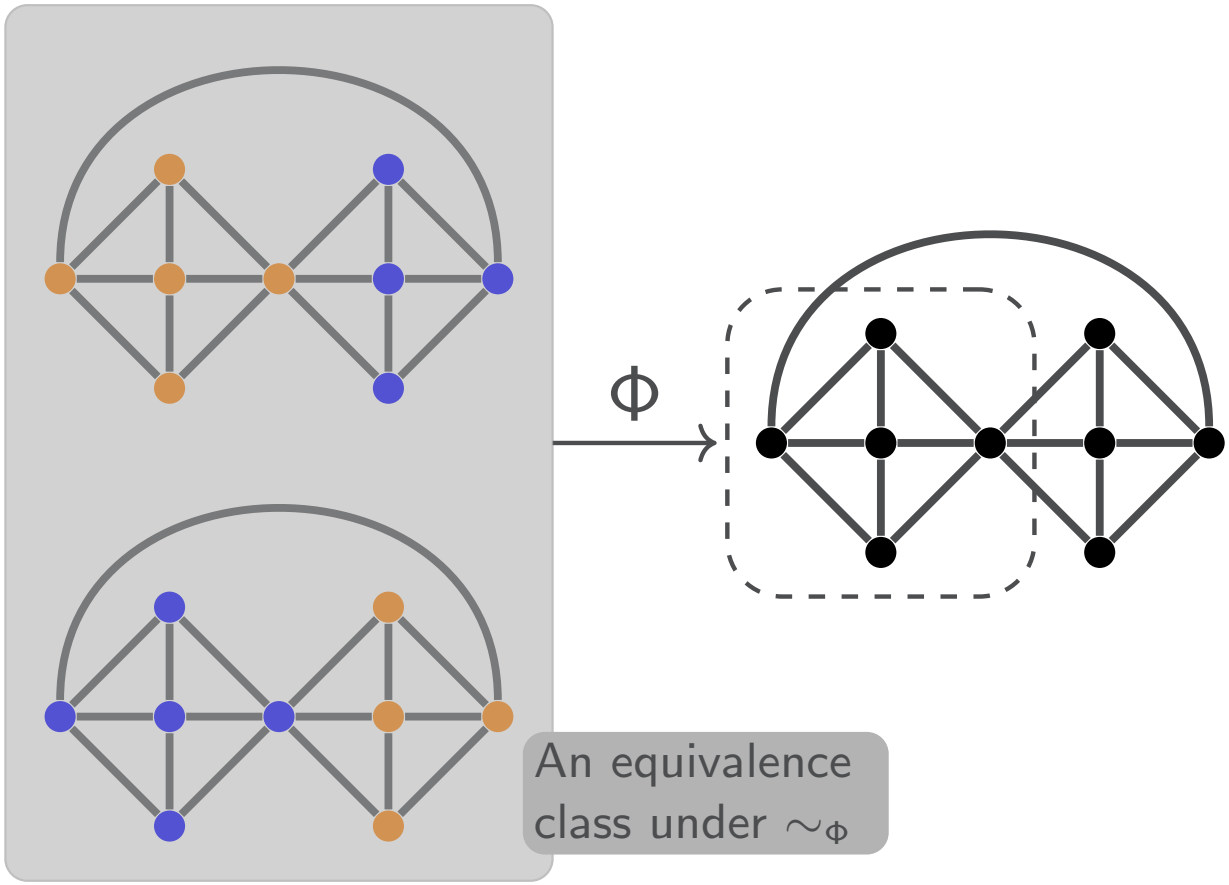
Let X_G be the set of all strategy profiles on G and let $\mathcal{Q}_G$ be the set of all vertex partitions of G. Define $\Phi : X_G \to \mathcal{Q}_G$



Partitioning Approach

Strategy Space

Let X_G be the set of all strategy profiles on G and let $\mathcal{Q}_G$ be the set of all vertex partitions of G. Define $\Phi : X_G \to \mathcal{Q}_G$



Reducing the State Space

Reduced Strategy Space

Define $\mathcal{A}_G$ which is the set of equivalence classes of X_G under Φ.

$$\mathcal{A}_G = X_G / S_m$$

Equilibrium partition

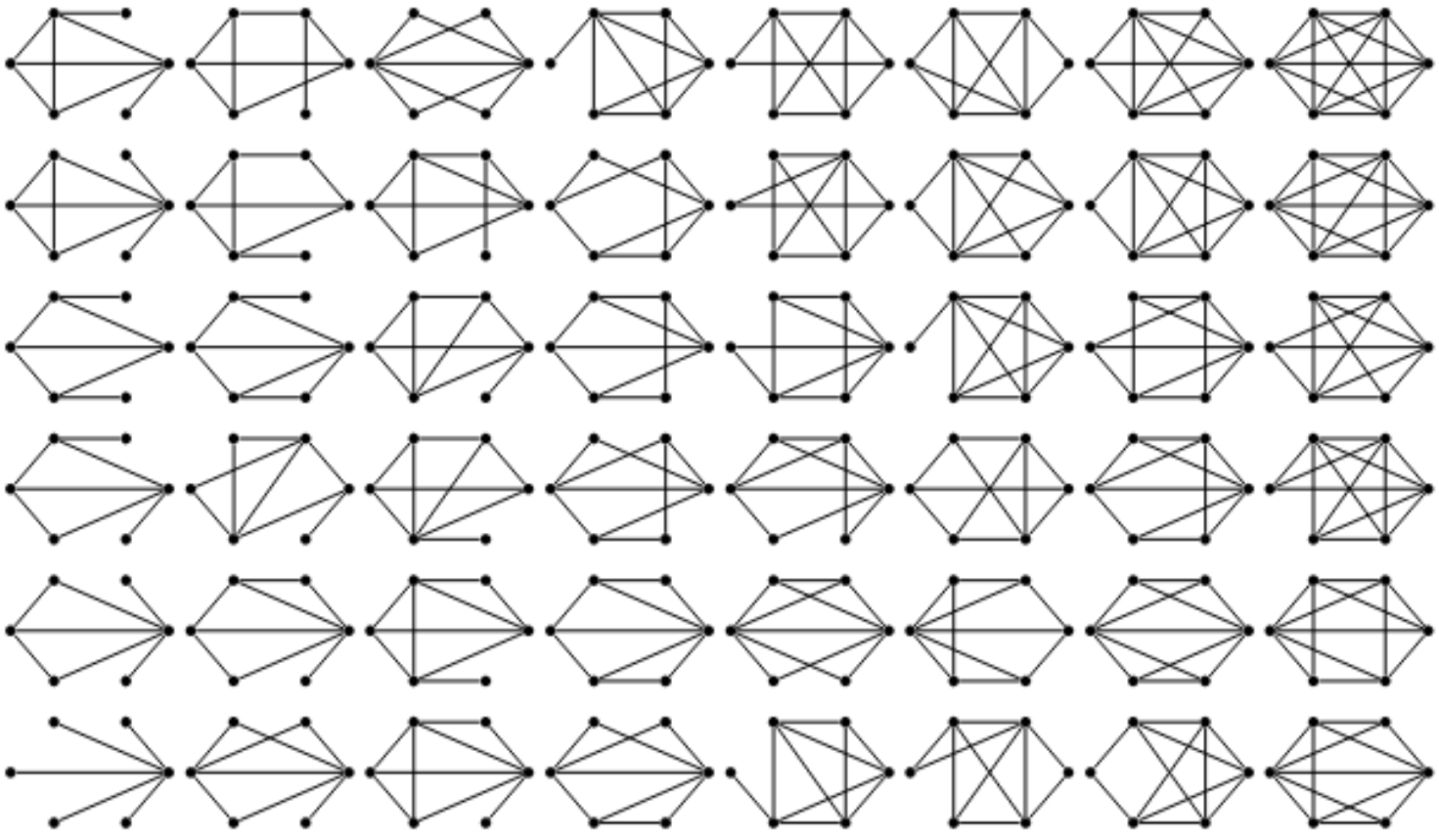
An equilibrium partition is a partition P such that $\Phi^{-1}P \in \mathcal{A}$ is a Nash equilibrium

Catalogue

The reduction in strategy space makes it possible to catalogue all of the equilibrium partitions on small graphs up to isomorphism

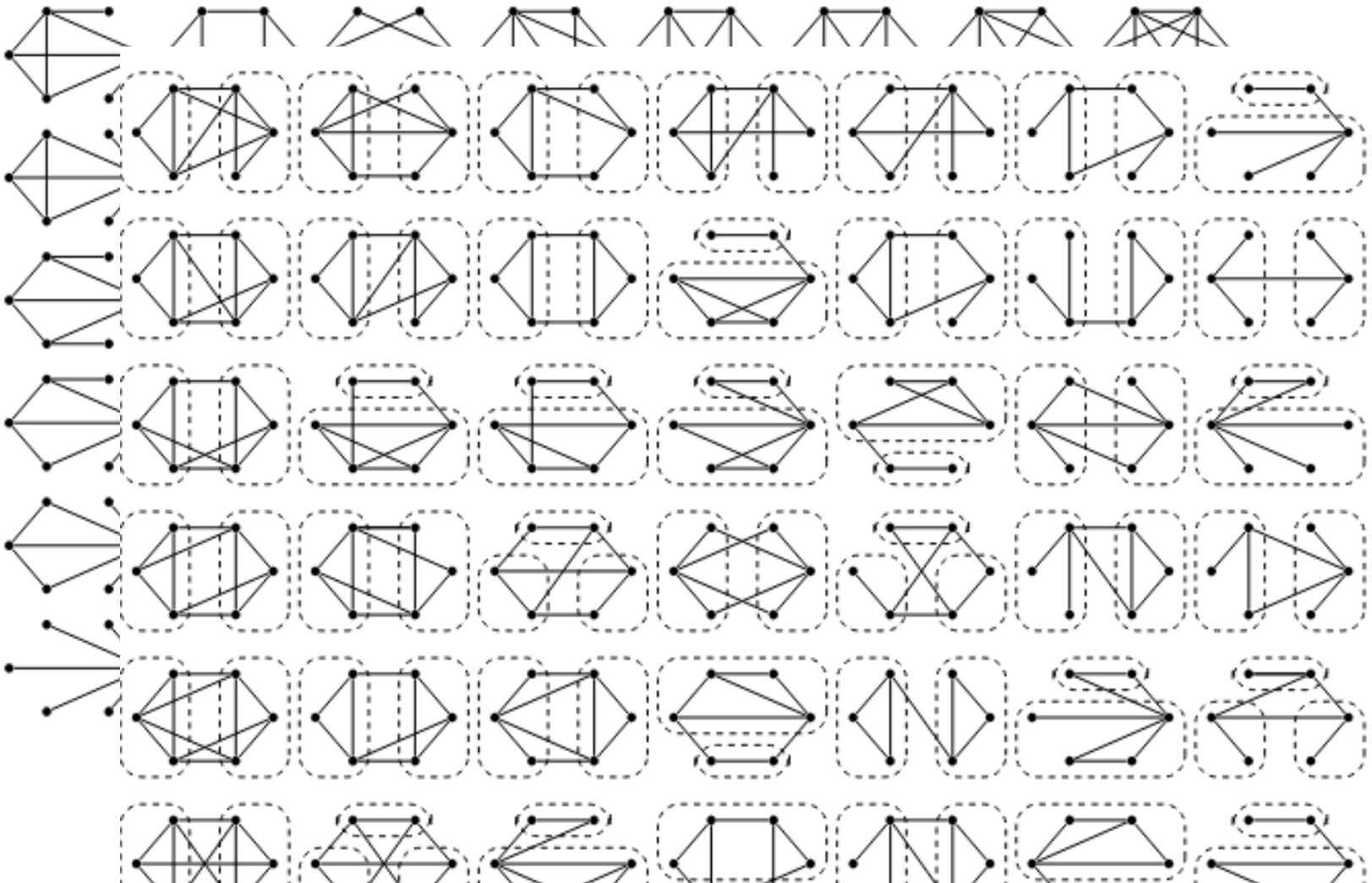
Catalogue

The reduction in strategy space makes it possible to catalogue all of the equilibrium partitions on small graphs up to isomorphism



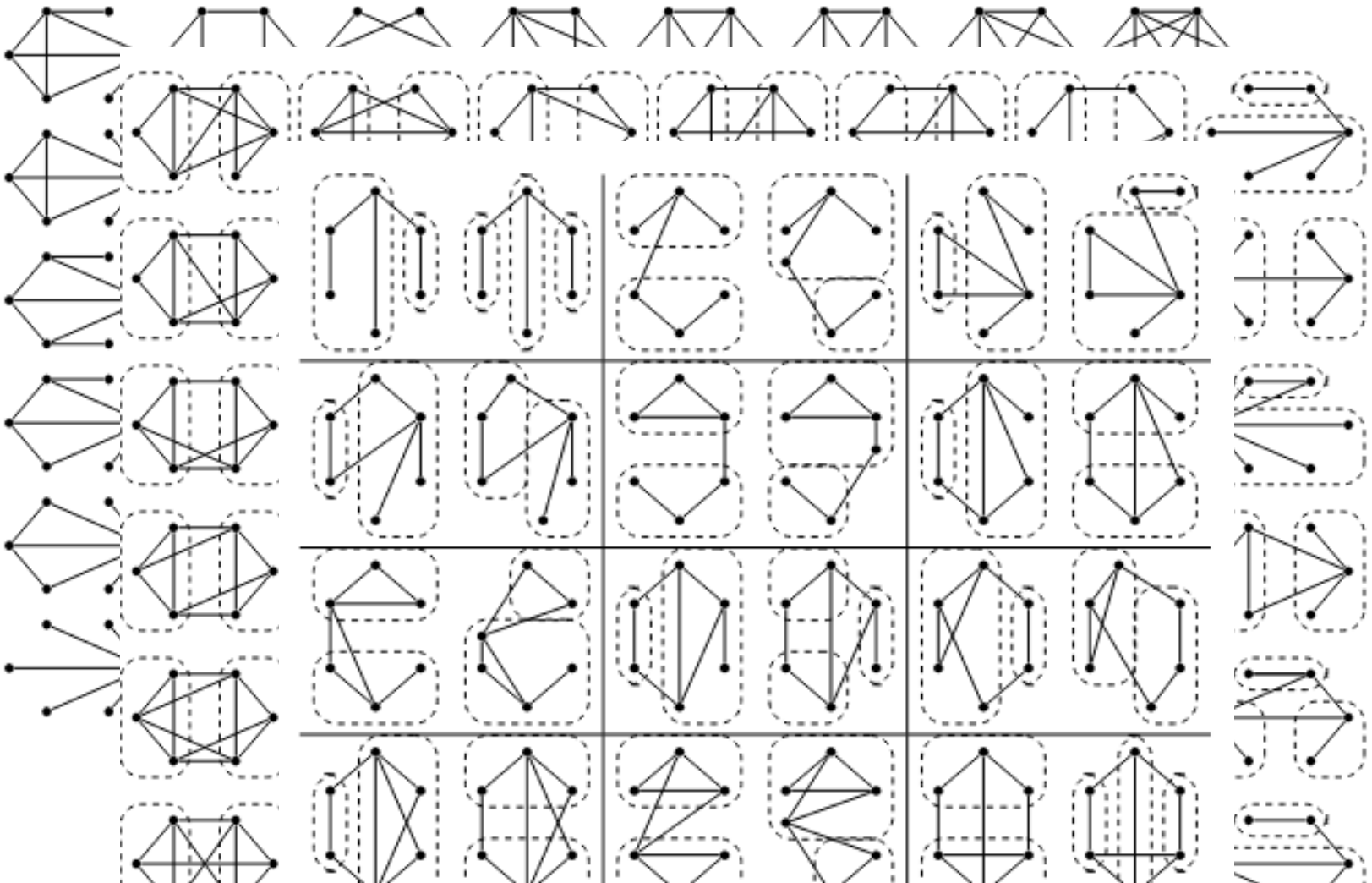
Catalogue

The reduction in strategy space makes it possible to catalogue all of the equilibrium partitions on small graphs up to isomorphism



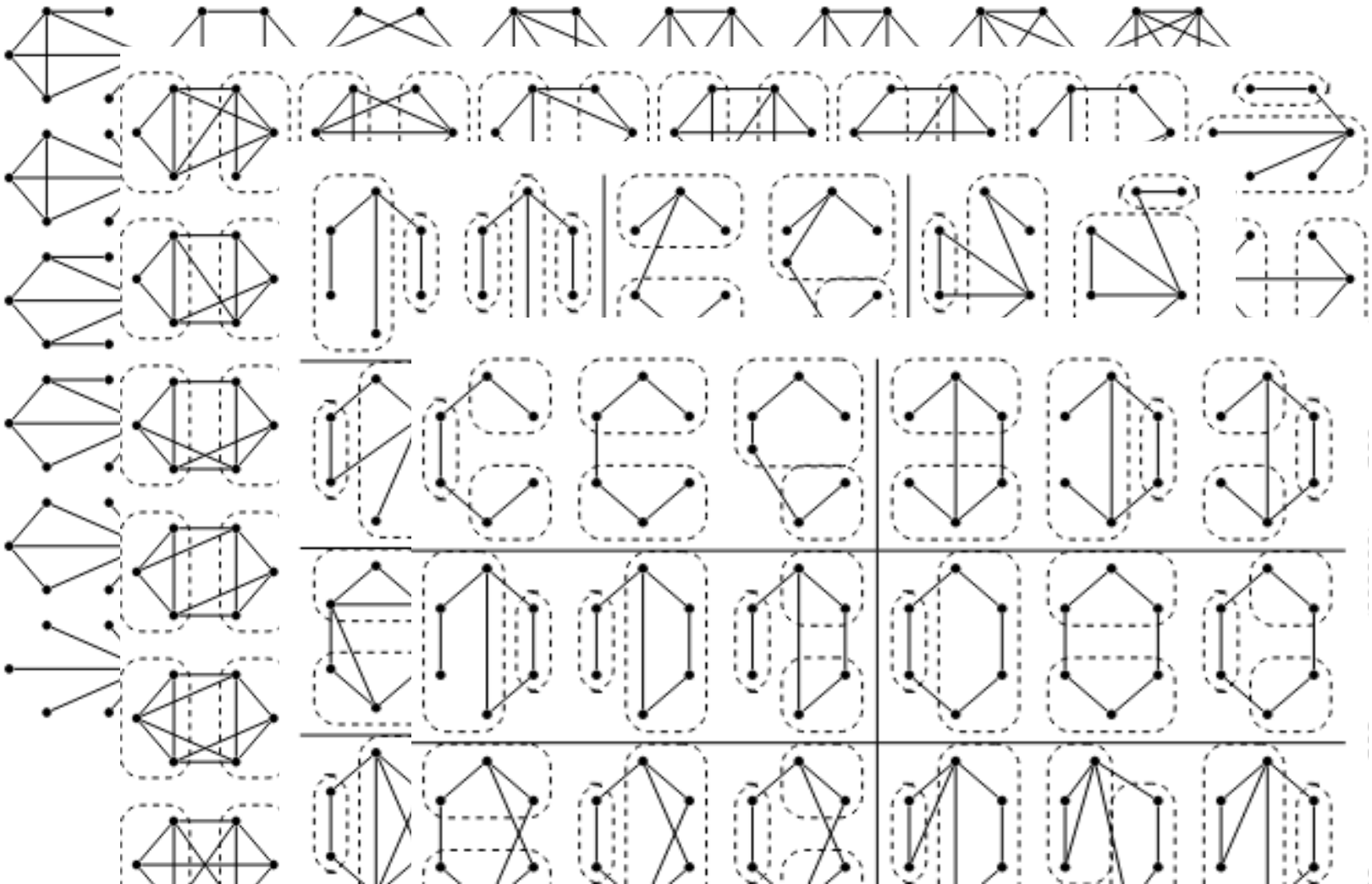
Catalogue

The reduction in strategy space makes it possible to catalogue all of the equilibrium partitions on small graphs up to isomorphism



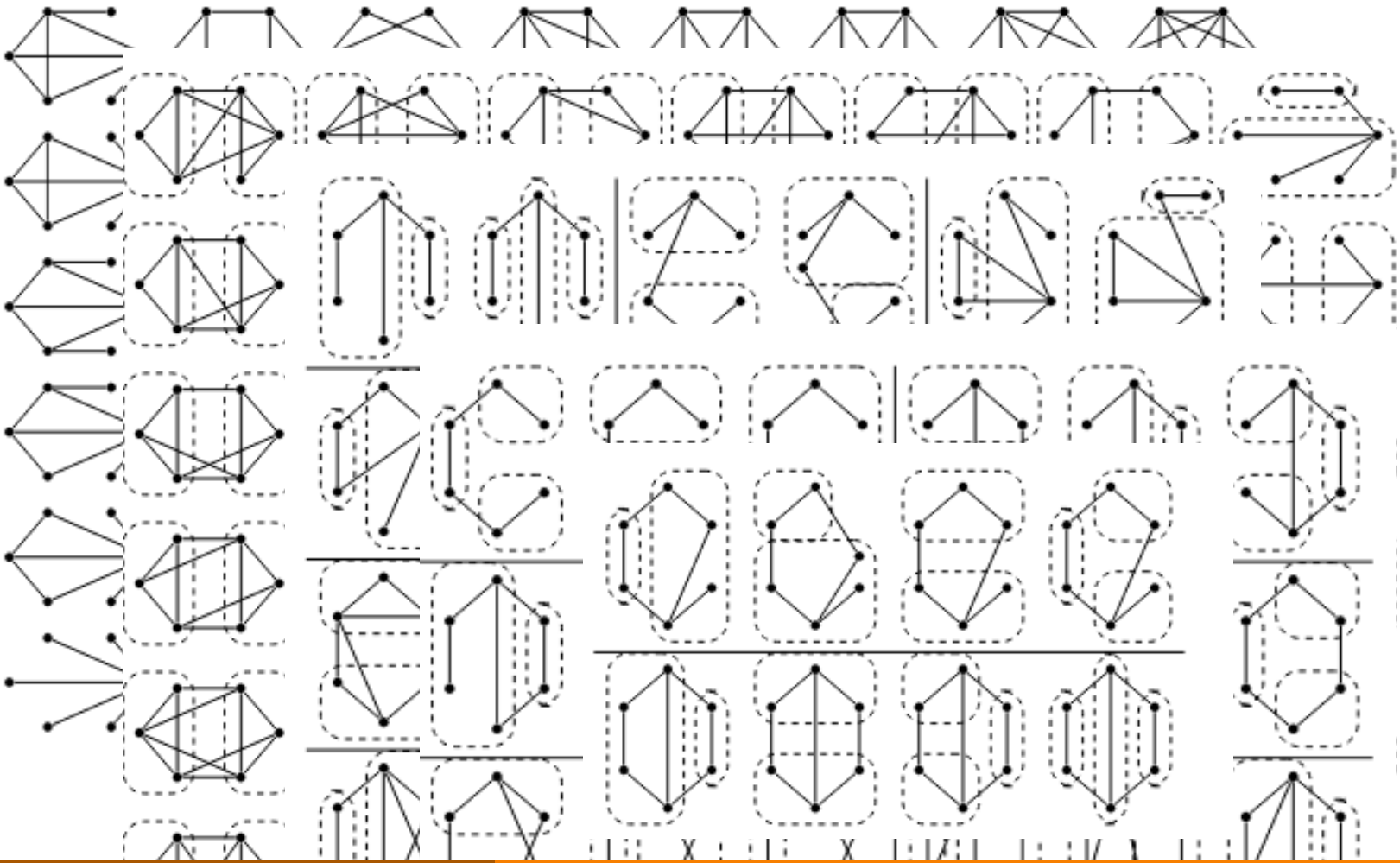
Catalogue

The reduction in strategy space makes it possible to catalogue all of the equilibrium partitions on small graphs up to isomorphism



Catalogue

The reduction in strategy space makes it possible to catalogue all of the equilibrium partitions on small graphs up to isomorphism



Simulations for dynamic

Simulating an Orbit

An orbit is a sequence of strategy profiles which satisfy

$$u(t+1) \in \left[\underset{c \in C}{\operatorname{argmax}}\, w_v(c|u(t)) \right]_{v \in V}$$

This is not unique and it does not necessarily converge to some limiting behavior.

The question is now: How likely is it that an orbit converges to a particular equilibrium partition?

Simulations for dynamic

Simulating an Orbit

An orbit is a sequence of strategy profiles which satisfy

$$u(t+1) \in \left[\underset{c \in C}{\operatorname{argmax}}\, w_v(c|u(t)) \right]_{v \in V}$$

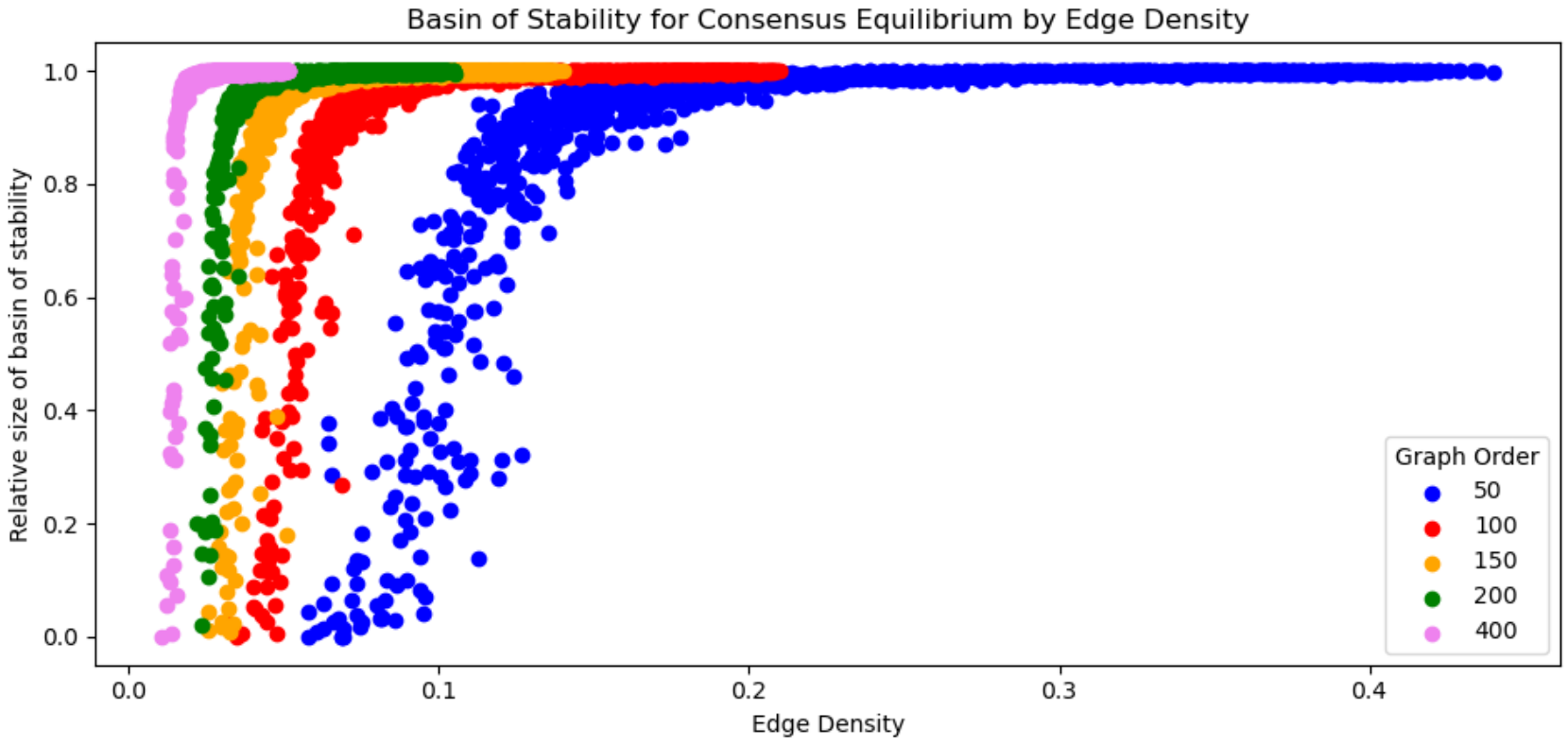
This is not unique and it does not necessarily converge to some limiting behavior.

The question is now: How likely is it that an orbit converges to a particular equilibrium partition?

Size of the Basin of Stability

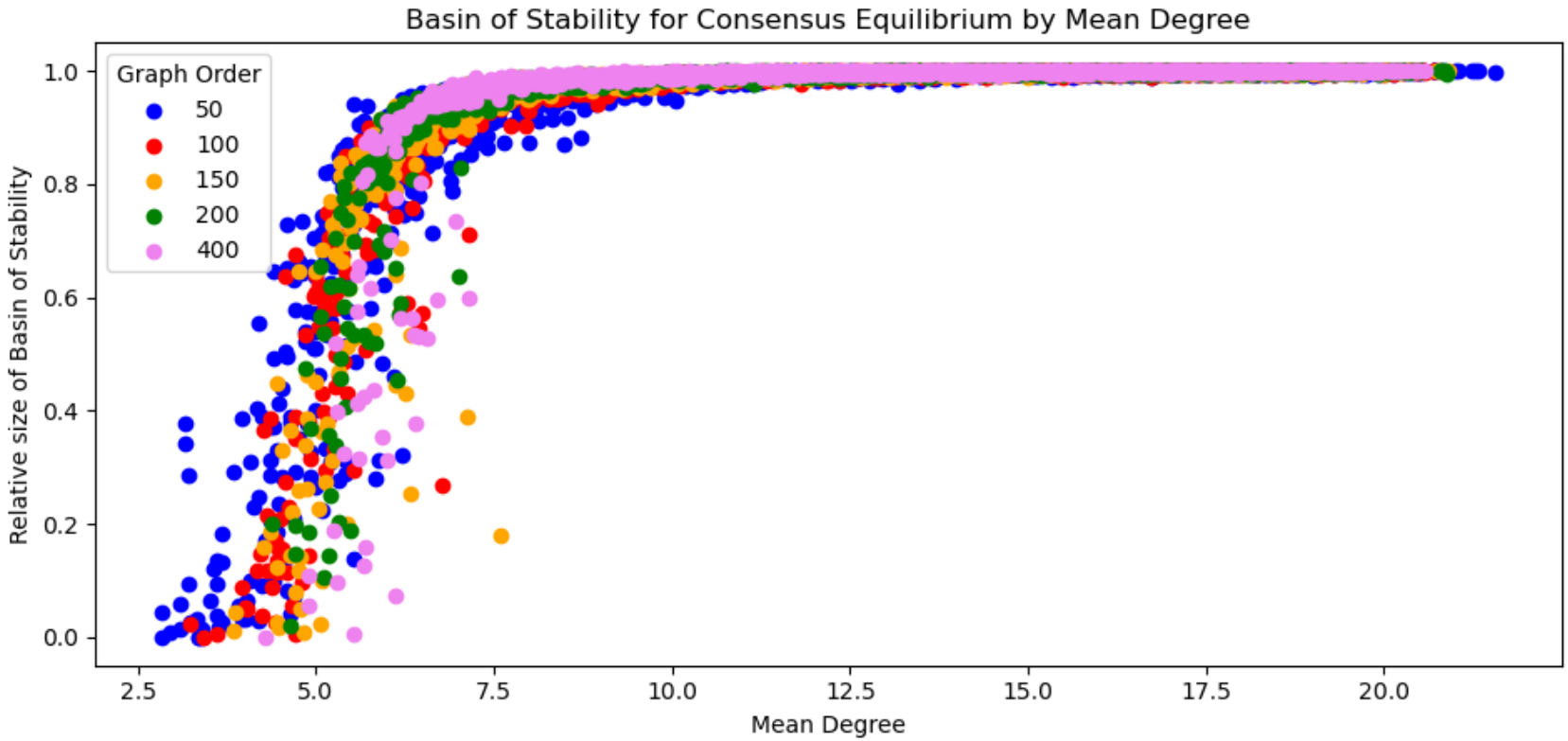
let $\varphi : \mathcal{Q}_G \to [0,1]$ be the function which takes partitions of G and gives the probability that an arbitrarily chosen initial condition will flow to that partition.

Basin of Stability for Consensus Equilibria



The more connected a graph is the more likely it is to converge to consensus.

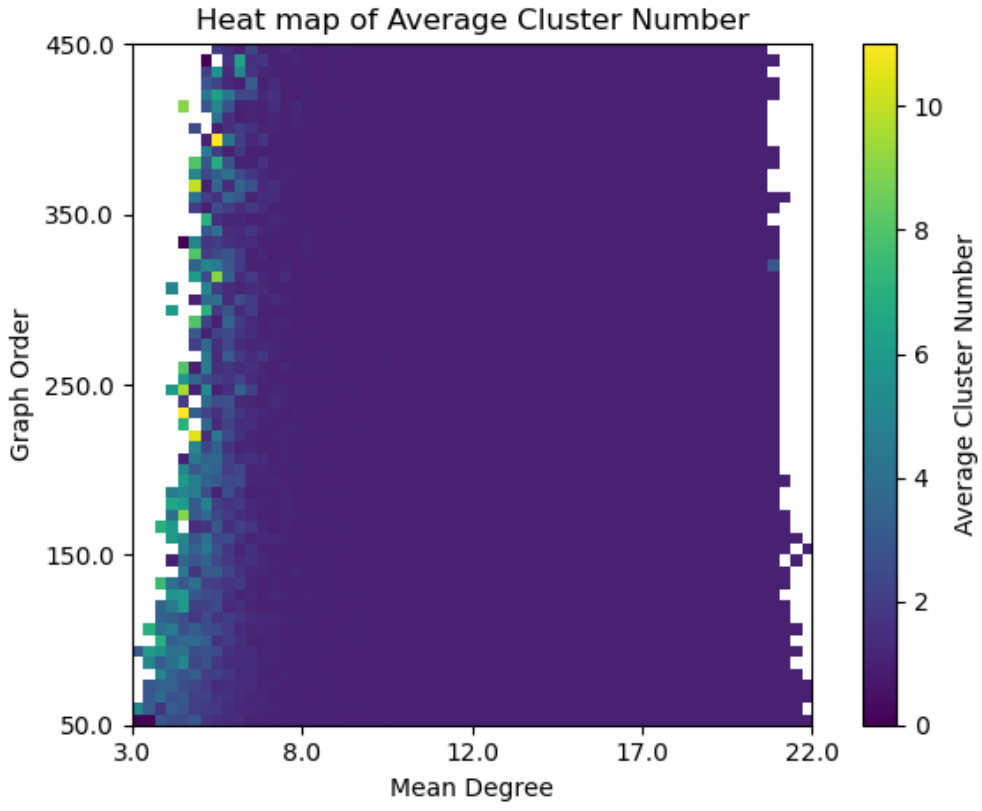
Basin of Stability for Consensus Equilibria



The more connected a graph is the more likely it is to converge to consensus.

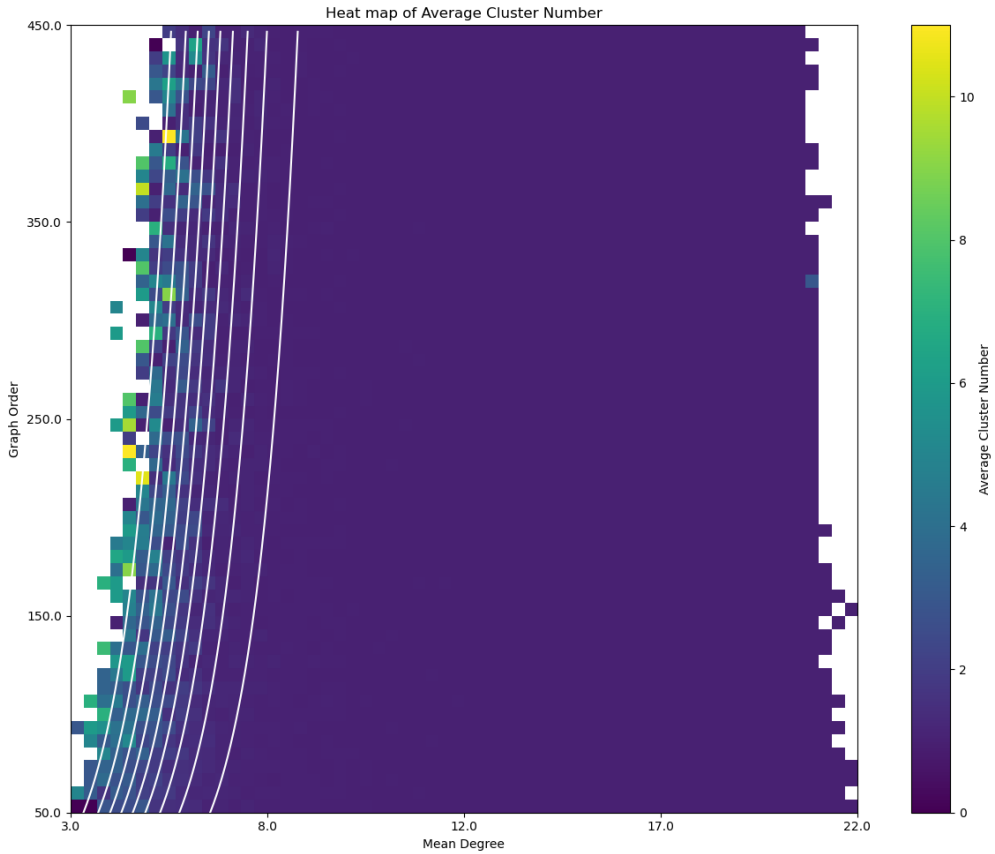
Observations from the simulation

Non-consensus equilibria only appear in the transitional region of connectedness in the E-R sense.



Observations from the simulation

Non-consensus equilibria only appear in the transitional region of connectedness in the E-R sense.



Results of the simulation

	Simulation
Main Achievement	
Numerical Speed	
Analytical Tractability	
Model Accuracy	

Results of the simulation

	Simulation
Main Achievement	Observation
Numerical Speed	
Analytical Tractability	
Model Accuracy	

Results of the simulation

	Simulation
Main Achievement	Observation
Numerical Speed	✓
Analytical Tractability	
Model Accuracy	

Results of the simulation

	Simulation
Main Achievement	Observation
Numerical Speed	✓
Analytical Tractability	X
Model Accuracy	

Results of the simulation

	Simulation
Main Achievement	Observation
Numerical Speed	✓
Analytical Tractability	X
Model Accuracy	✓

Results of the simulation

	Simulation
Main Achievement	Observation
Numerical Speed	✓
Analytical Tractability	X
Model Accuracy	✓ ✓

Results of the simulation

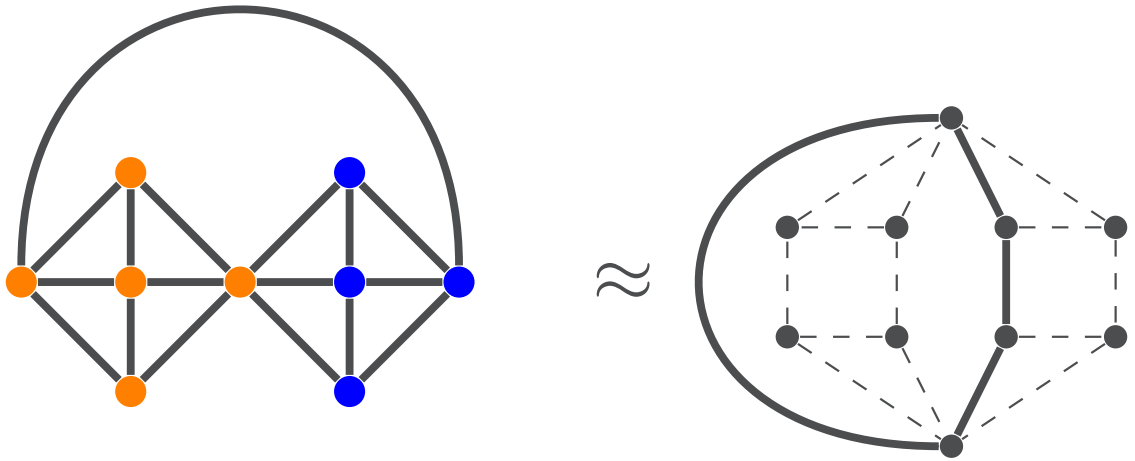
	Simulation
Main Achievement	Observation
Numerical Speed	✓
Analytical Tractability	X
Model Accuracy	✓ ✓

Main result

Non-consensus equilibria occur in the transitional area of connectedness, when graphs are "minimally connected."

Minimal Subgraphs

To try to achieve more analytical power we will attempt to consider the structured coordination game in the dual sense.



Potential Games

The Coordination game is a potential game meaning there is a function $\mathcal{W}$ such that for two strategy profiles $u = (u_v, u_{-v})$ and $u' = (u'_v, u_{-v})$ then

$$\mathcal{W}(u) - \mathcal{W}(u') = w_v(u_v|u) - w_v(u'_v|u')$$

Lemma 2

$\mathcal{W}(u) = \frac{-1}{2}E(\Phi u)$ is a potential function for the pure coordination game where $E(P)$ is the size of the cutset for the partition P.

Potential Games

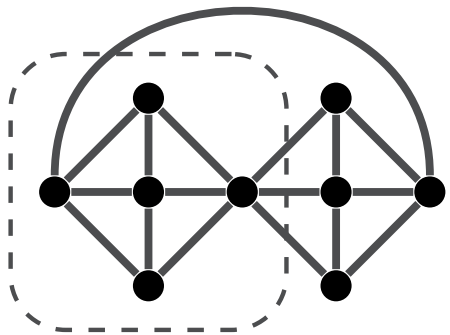
The Coordination game is a potential game meaning there is a function $\mathcal{W}$ such that for two strategy profiles $u = (u_v, u_{-v})$ and $u' = (u'_v, u_{-v})$ then

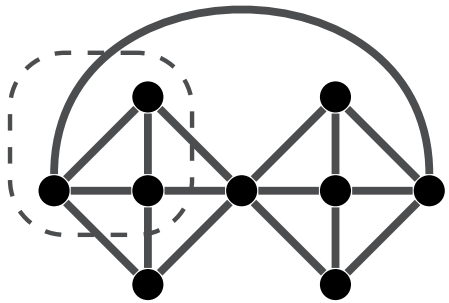
$$\mathcal{W}(u) - \mathcal{W}(u') = w_v(u_v|u) - w_v(u'_v|u')$$

Lemma 2

$\mathcal{W}(u) = \frac{-1}{2}E(\Phi u)$ is a potential function for the pure coordination game where $E(P)$ is the size of the cutset for the partition P.

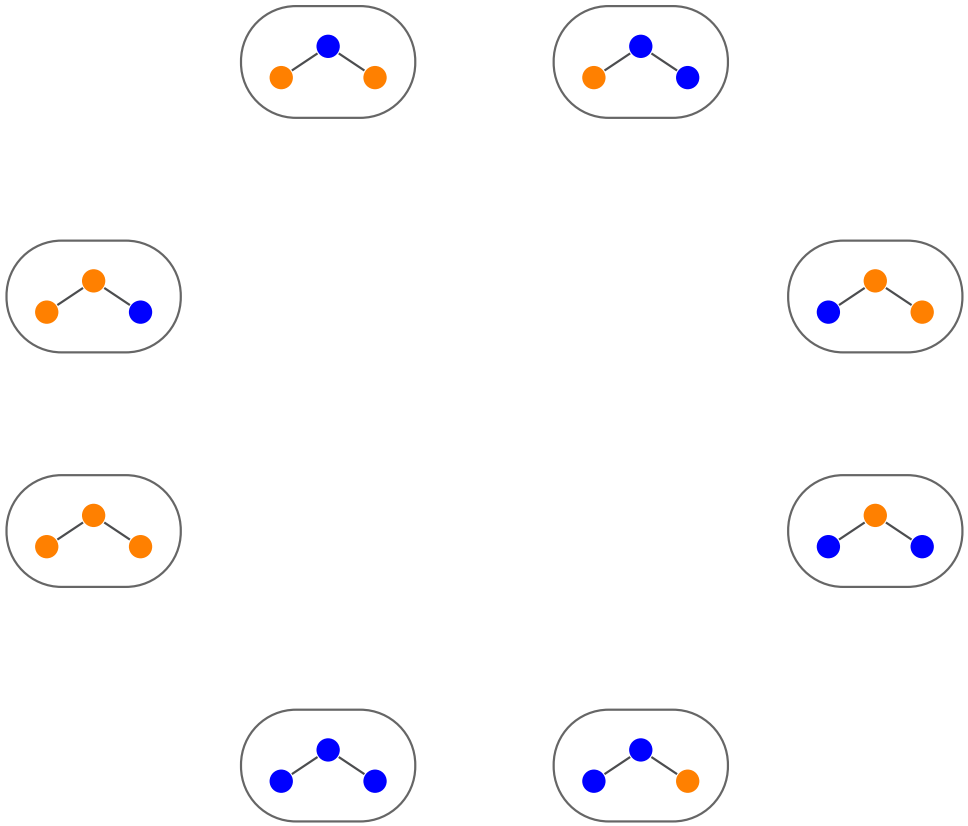
$\mathcal{W}(u) = -2$





$\mathcal{W}(u) = \frac{-5}{2}$

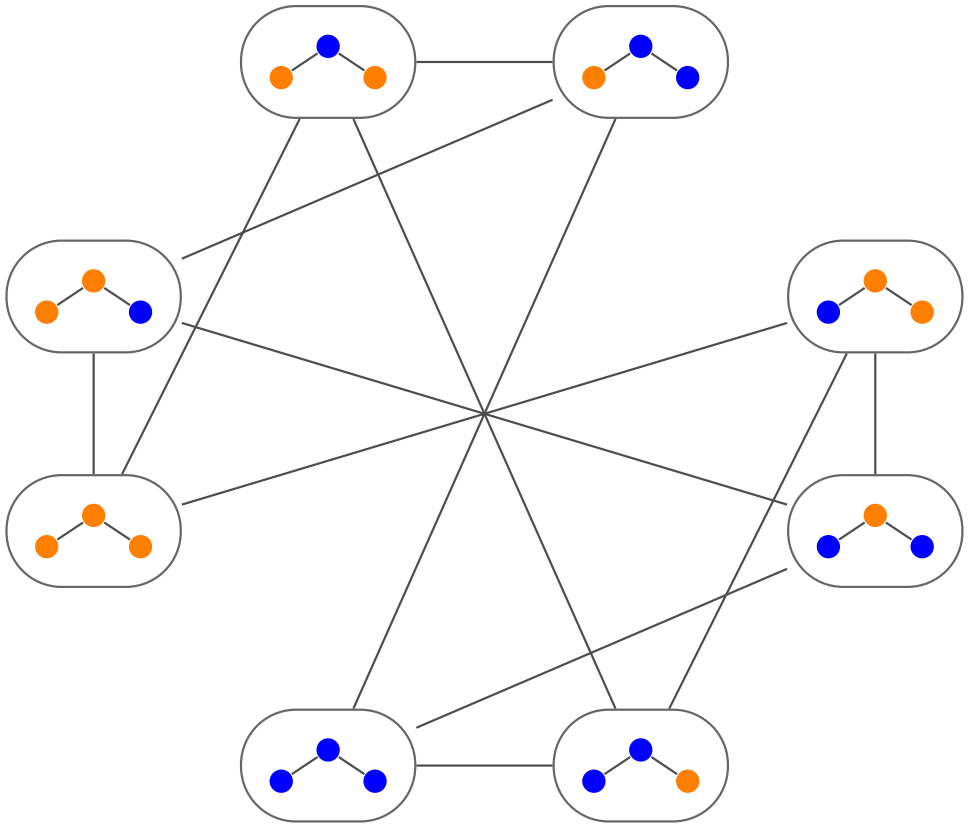
Defining locality



X_{P_3}

The set of all strategy profiles on the graph P_3 (here only the strategy profiles with 2 strategies are pictured).

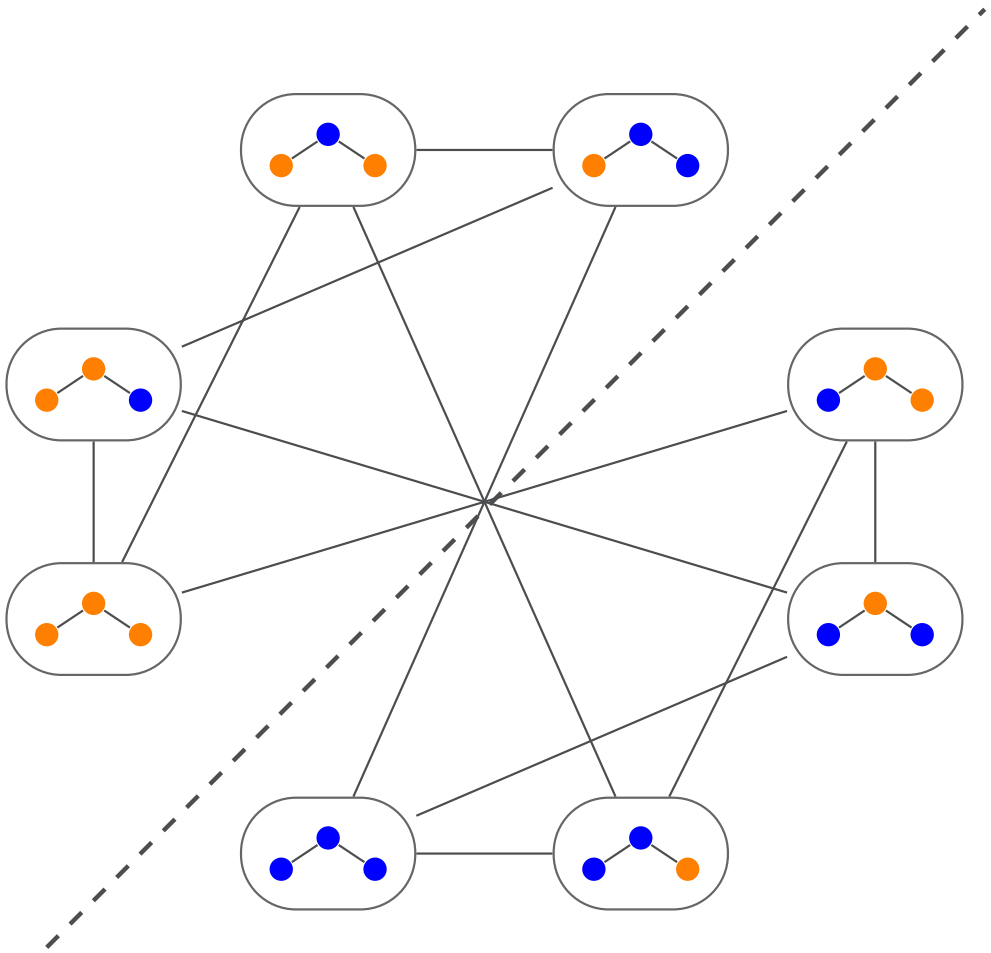
Defining locality



(X_{P_3}, d)

Two strategy profiles are adjacent if they are separated by a single strategic change. This adjacency defines a metric d.

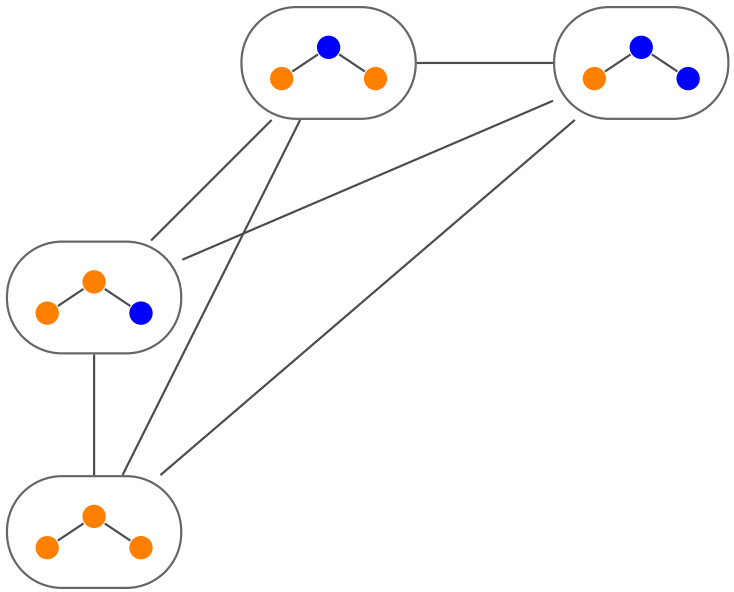
Defining locality



(X_{P_3}, d)

Two strategy profiles are adjacent if they are separated by a single strategic change. This adjacency defines a metric d.

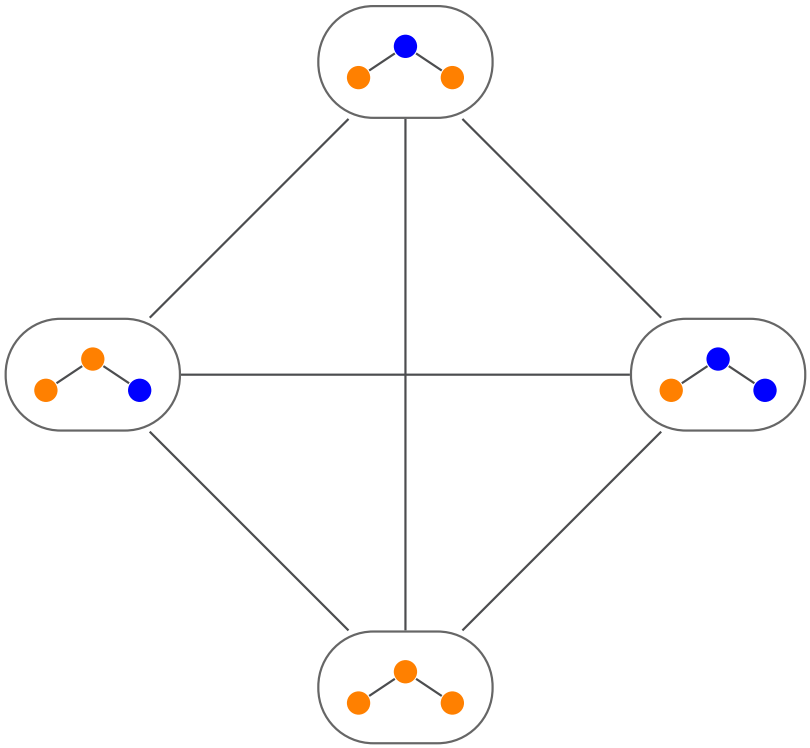
Defining locality



$(\mathcal{A}_{P_3}, d)$

In the same way as before, we reduce the metric space to equivalence classes under $\sim_\Phi$. Each edge is a single strategic change and (maybe) a strategic permutation

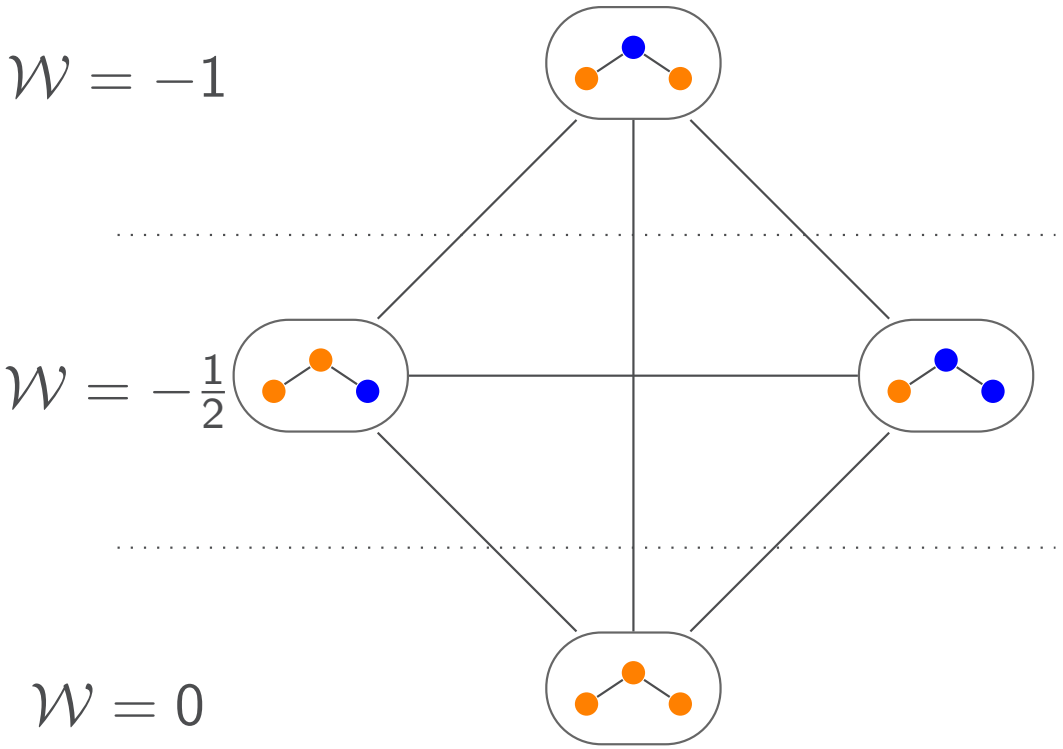
Defining locality



$(\mathcal{A}_{P_3}, d)$

In the same way as before, we reduce the metric space to equivalence classes under $\sim_\Phi$. Each edge is a single strategic change and (maybe) a strategic permutation

Defining locality



Each element of $\mathcal{A}$ has an associated game potential $\mathcal{W}(u)$. If u maximizes $\mathcal{W}(u)$ locally in $(\mathcal{A}_G, d)$ it is a Nash equilibrium

Finding Local Maxima

Let $\Gamma_{\mathcal{A}}(u) := \{a \in \mathcal{A}; d(a,u) \leq 1\}$. With this neighborhood defined we can say that

Corollary 3.1

If $\mathcal{W}(u) \geq \mathcal{W}(u')$ for all $u' \in \Gamma_{\mathcal{A}}(u)$ then u is a Nash equilibrium.

Because maximizing $\mathcal{W}$ is the same as minimizing the size of the cut set we can say that Nash equilibria of the pure coordination game on the graph G are local minimizers of the size of the cutsets they induce.

Model Setup

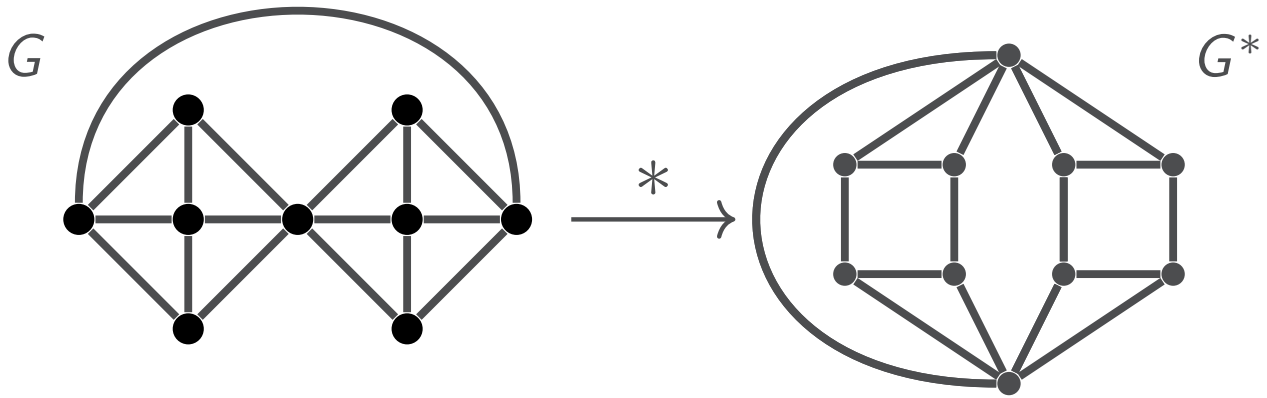
Unfortunately $\mathcal{A}$ is not an easy set to understand or imagine. In this chapter I will show that the metric space is isometric (or quasiisometric) to a different space that is easier to imagine.

Graphical Dual

Dual graph

Let G be a 3-edge connected planar graph the dual G^* is a simple planar graph.

- Every face in G is a vertex of G^*
- An edge separating two faces in G corresponds to an edge between the two corresponding vertices in G^*
- Thus, every vertex in G corresponds to a face of G^*.



Subgraph Metric Space

Call the set of bridge free subgraphs of G, $\mathcal{SC}_G$. We say that two subgraphs of $\mathcal{SC}_G$ are adjacent in the quasimetric D if and only if they are separated by a Single Face Rewiring (SFR).

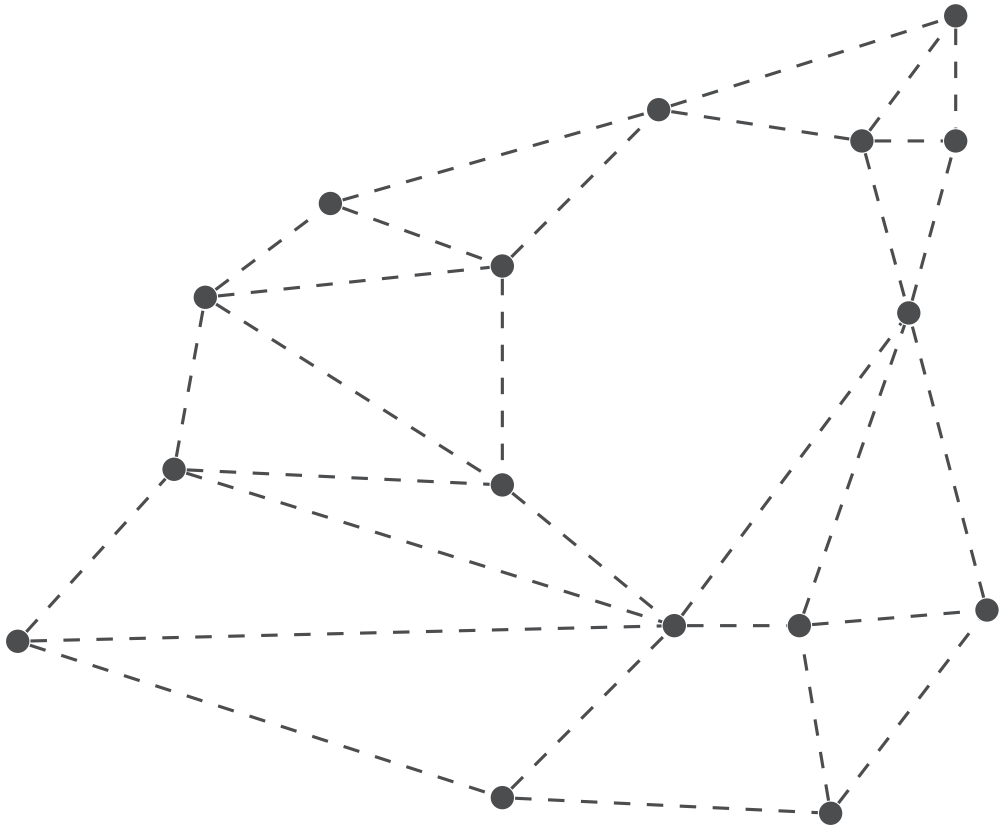
Single Face Rewiring (SFR)

For a simple planar graph G^* and a subgraph $s \in \mathcal{SC}_{G^*}$, a Single Face Rewiring from s to r is a resampling of edges from G^* so that, for one face f of G^*

1. **Locality:** Any edge not incident to the face f which was in s is still in r after the *SFR*.
2. **Closure:** Edges incident to f cannot be added in such a way that would create a bridge in r.
3. **Connectivity:** If two departure vertices were connected by a path along f in s but are not connected by a path along f in r, then they must be entirely disconnected in r.

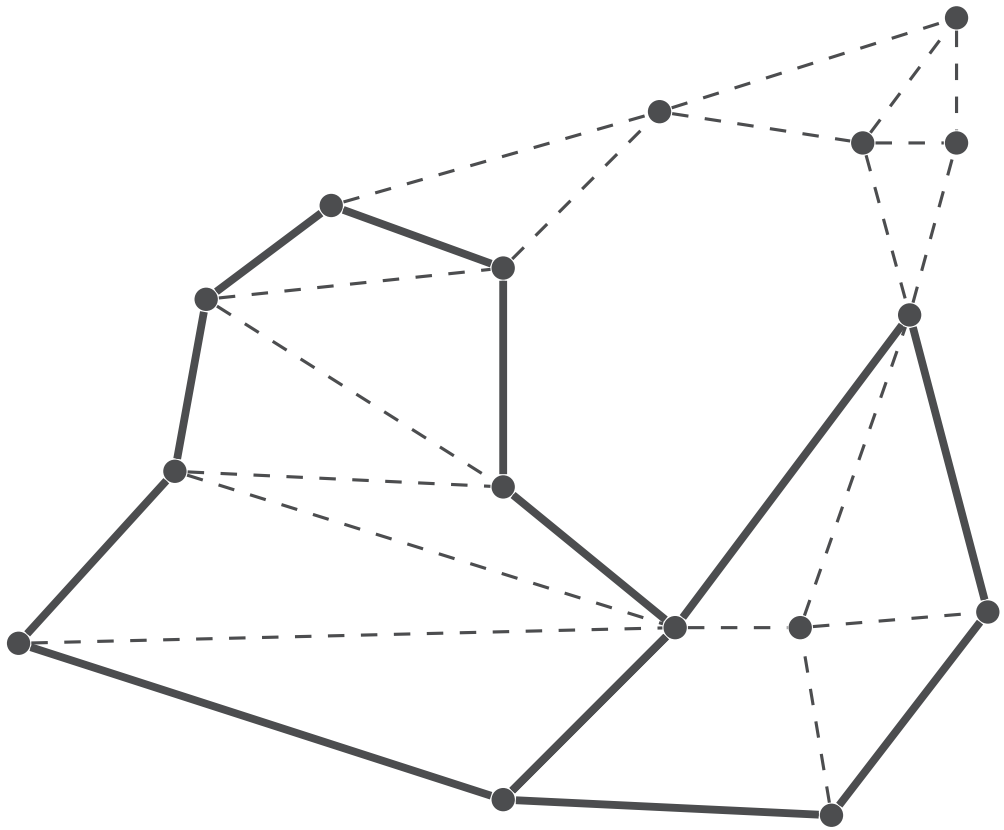
Example one

A graph G



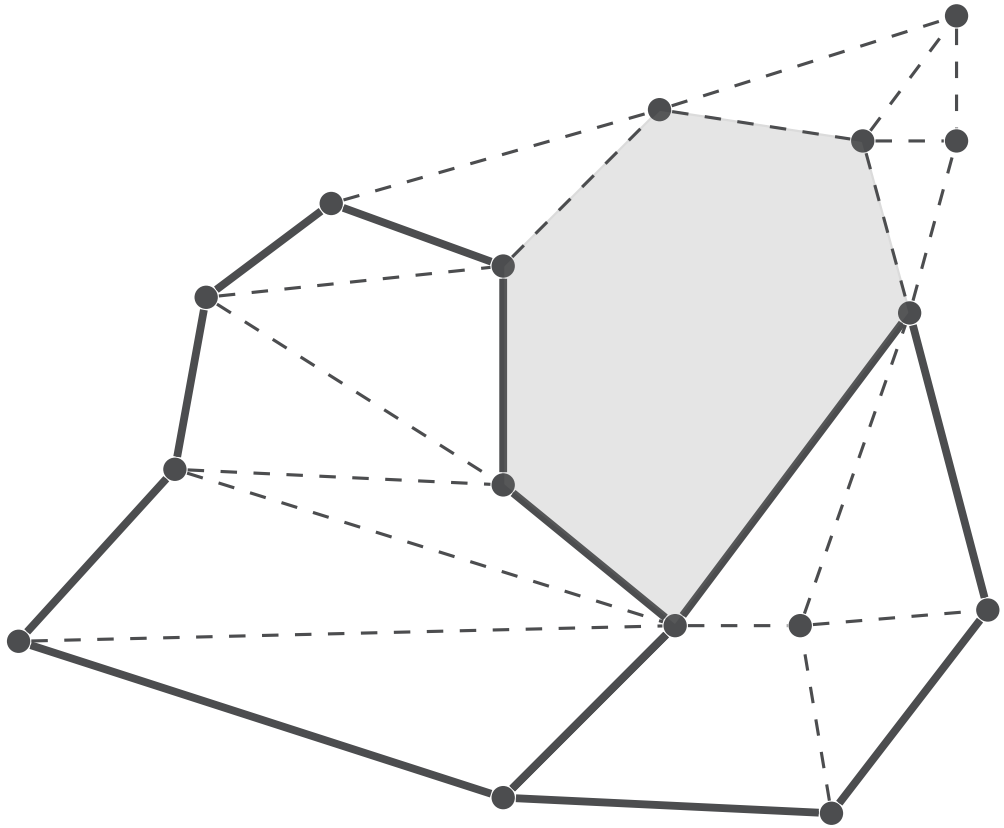
Example one

A bridge free subgraph $s \in \mathcal{SC}_G$



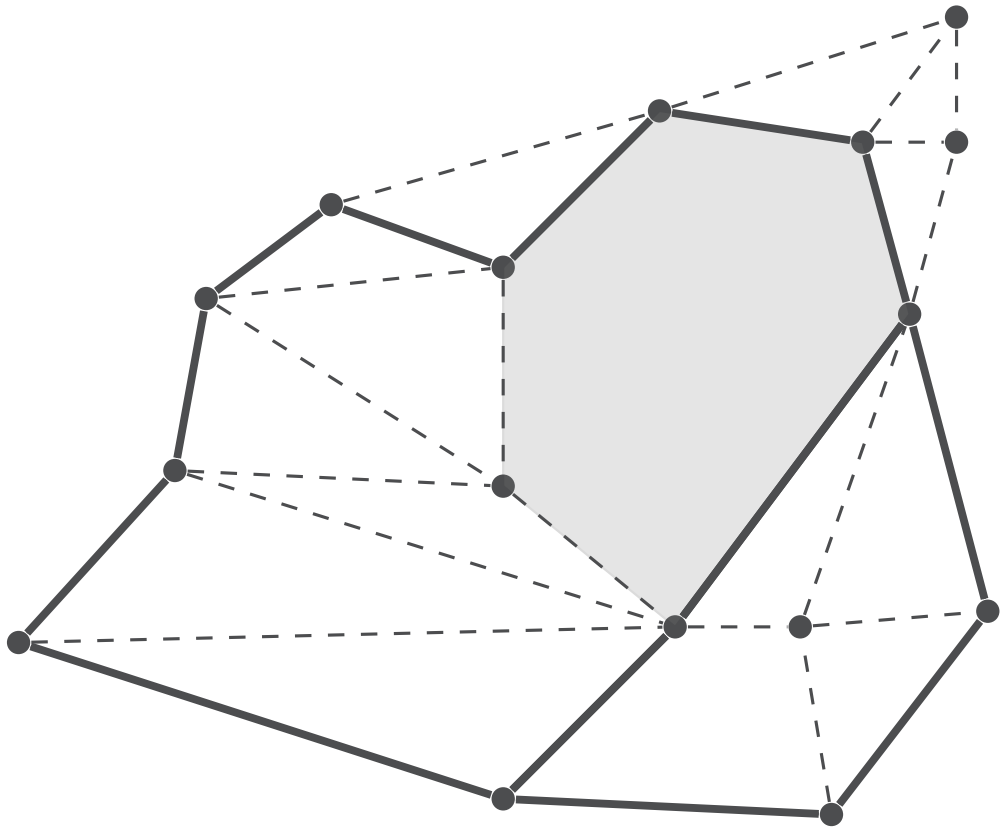
Example one

select a face f of G



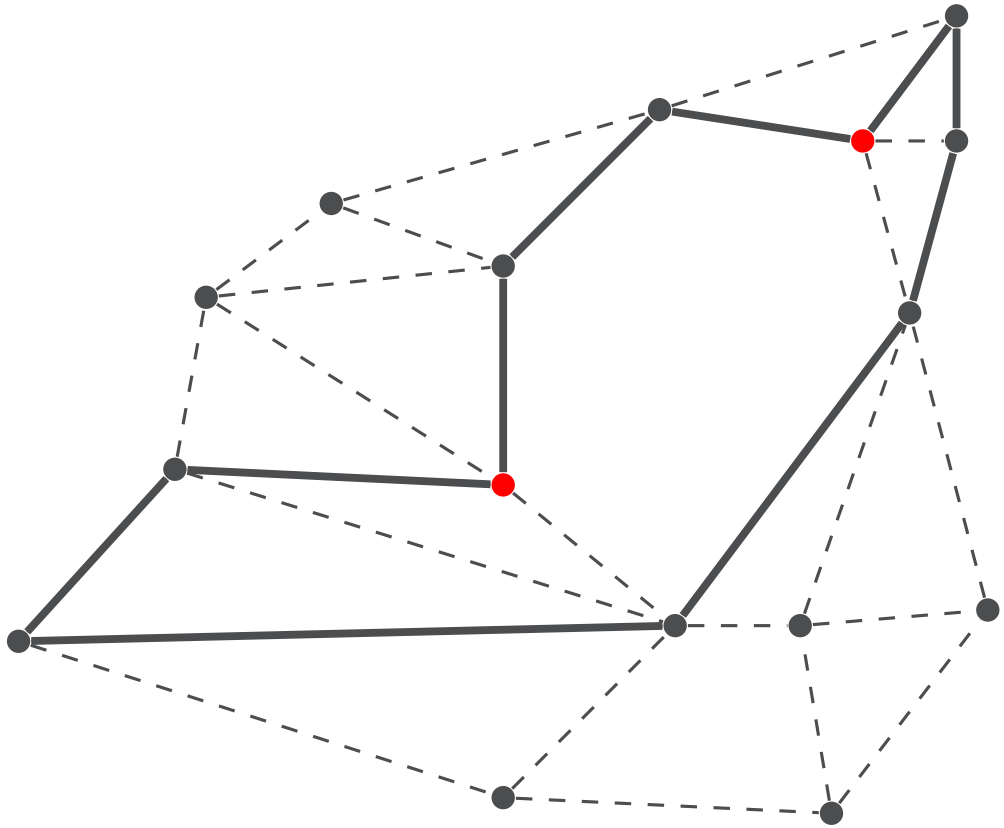
Example one

Resamples edges of G incident to f to form new subgraph $s' \in \mathcal{SC}_G$



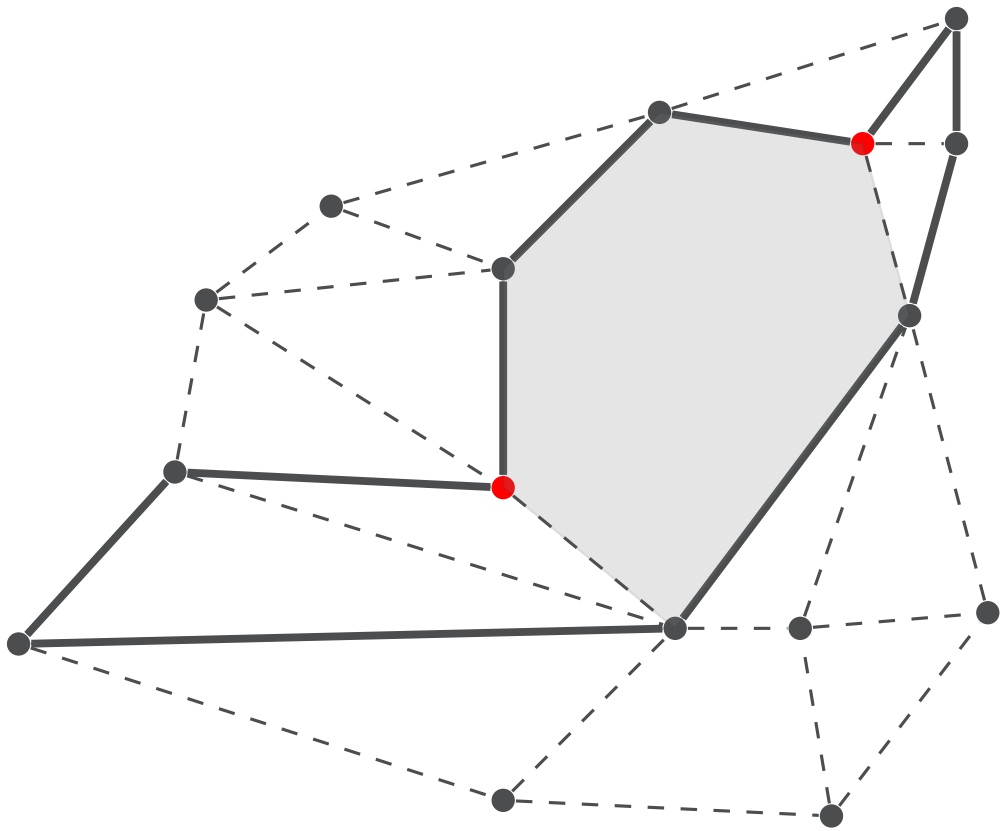
Example two

another subgraph $s \in \mathcal{SC}_G$



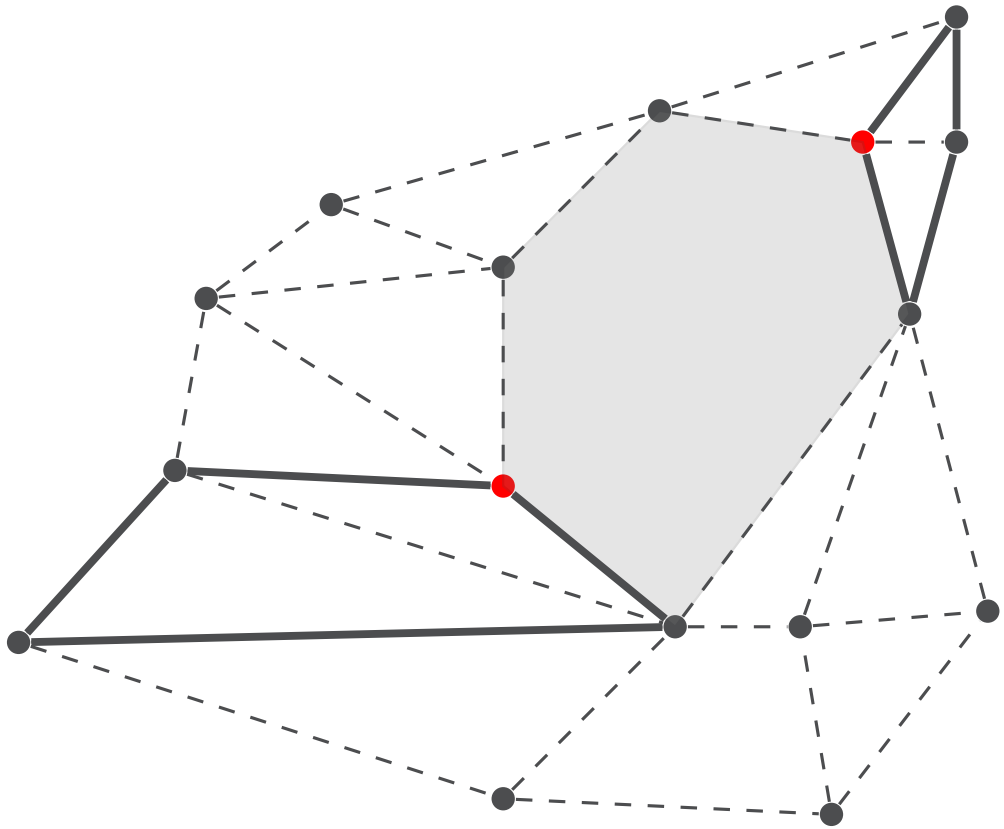
Example two

If two departure vertices are initially connected by a path along f in s...



Example two

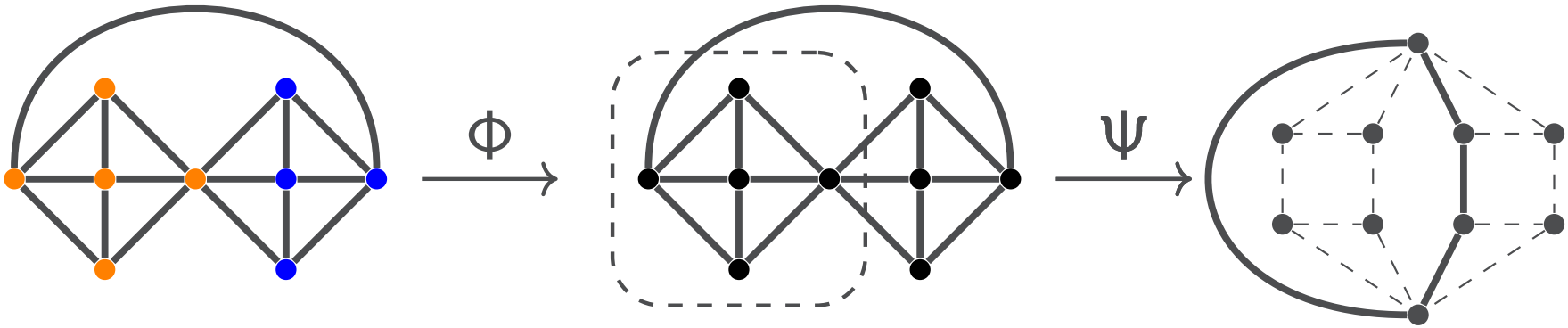
... and are not connected by a path along f in s' they must be entirely discon-nected in s'



Quasiisometry

$\Psi : \mathcal{Q}_G \to \mathcal{SC}_{G^*}$

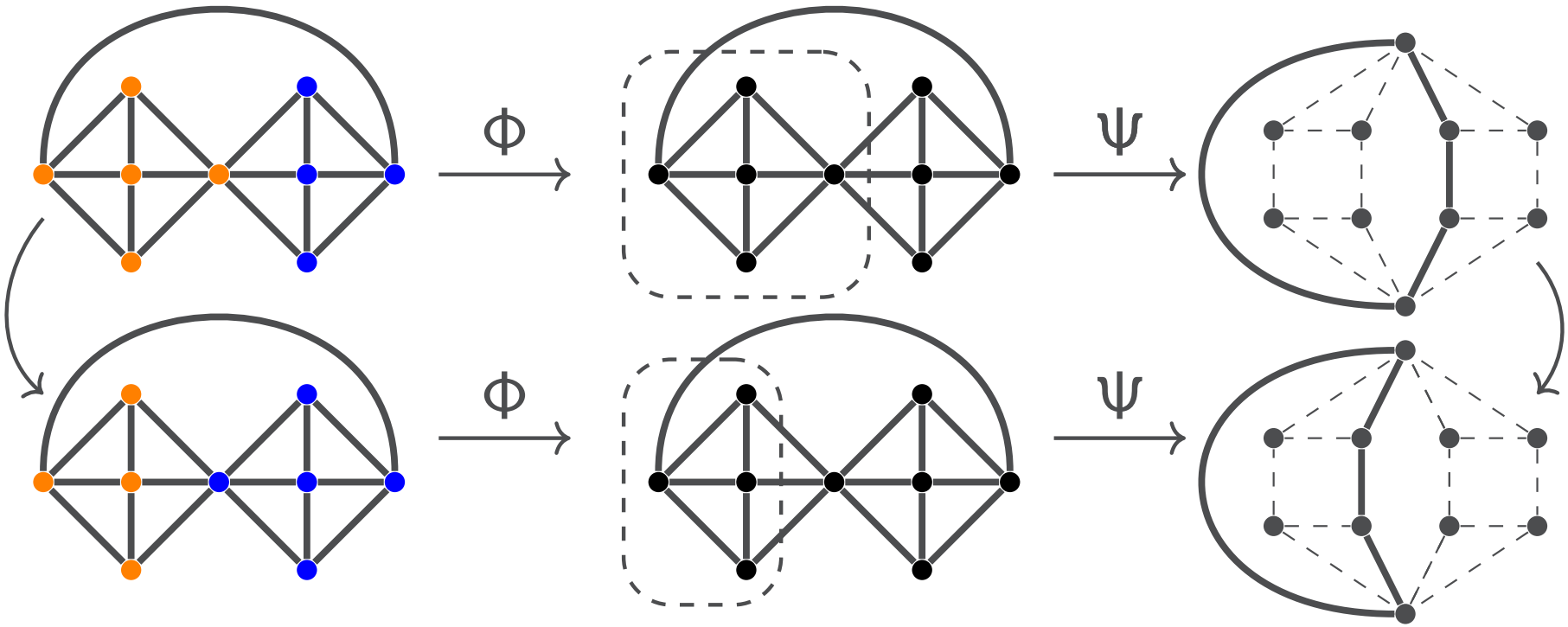
Let $\Psi : \mathcal{QC}_G \to \mathcal{SC}_{G^*}$ where if $e \in E(P)$ then $e^* \in \Psi(P)$.



Quasiisometry

Lemma 3.6

For a planar 3-edge connected graph G, if $u \in \mathcal{A}_G$ has $\Psi u \in \mathcal{QC}_G$ and $d(u, u') = 1$ in $(\mathcal{A}_G, d)$ for some u', then $D(\Psi \circ \Phi u, \Psi \circ \Phi u') = 1$ in $\mathcal{SC}_{G^*}$



Proof of the Quasiisometry

Assume that $d(u, u') = 1$ and that $\Phi u \in \mathcal{QC}_G$. We will show that there is a SFR between $s := \Psi \circ \Phi u$ and $s' := \Psi \circ \Phi u'$

Step 1: Locality

u and u' differ by a single strategic change, call the player that changed their strategy v and note that for every edge not incident to v, the players on either end of the edge did not change their strategy. If v corresponds to the face f in G^* consider an edge e^* not incident to f

$$e^* \in s \implies e \in E(\Phi u) \implies e \in E(\Phi u') \implies e^* \in s'$$

$$e^* \notin s \implies e \notin E(\Phi u) \implies e \notin E(\Phi u') \implies e^* \notin s'$$

Every edge of G^* not incident to f in s must also be present in s'

Proof of the Quasiisometry

Assume that $d(u, u') = 1$ and that $\Phi u \in \mathcal{QC}_G$. We will show that there is a SFR between $s := \Psi \circ \Phi u$ and $s' := \Psi \circ \Phi u'$

Step 2:Closure

For any partition P, observe that the cutset of P is the union of cutsets for the partitions $\{P^i, (P^i)^c\}$ for $i = 1, ..., m$. For any such partition into two parts,

$$E(\{P^i, (P^i)^c\}) \approx c \in C(G^*) \subset \mathcal{SC}_{G^*}$$

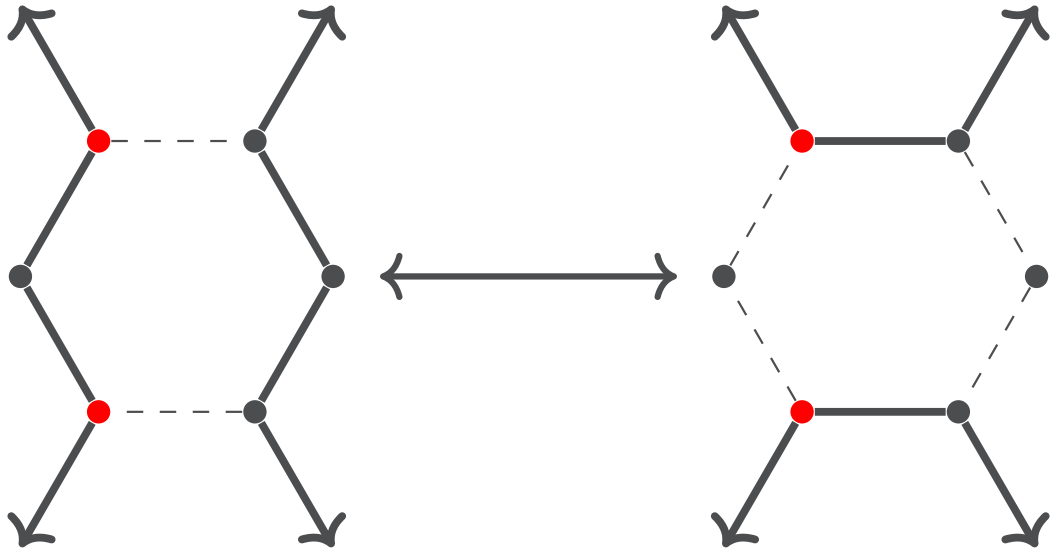
where $C(G^*)$ is the cycle space of G^*. Thus ΨP is the union of cycles and thus can never contain a bridge.
s and s' are therefore bridge free. (resampling the edges did not create a bridge in s').

Proof of the Quasiisometry

Assume that $d(u, u') = 1$ and that $\Phi u \in \mathcal{QC}_G$. We will show that there is a SFR between $s := \Psi \circ \Phi u$ and $s' := \Psi \circ \Phi u'$

Step 3: Connectivity

Suppose there are two departure vertices a and b of f which are connected by a path incident to f in s but not connected by a path incident to f in s'. In order to be an SFR, a and b must be disconnected in s'

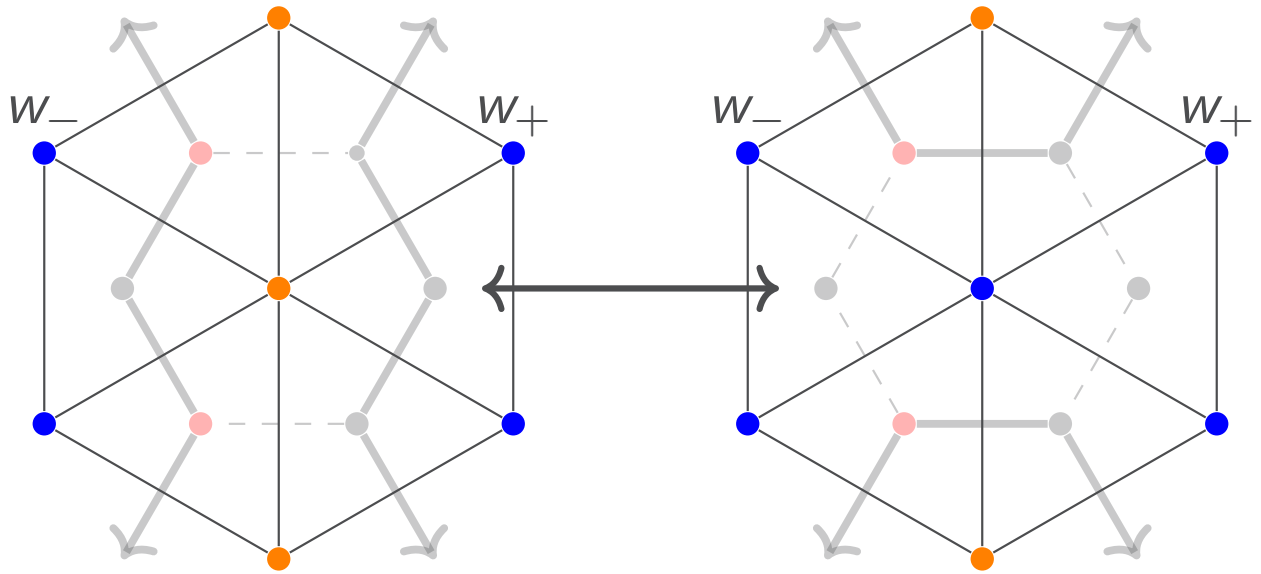


Proof of the Quasiisometry

Assume that $d(u, u') = 1$ and that $\Phi u \in \mathcal{QC}_G$. We will show that there is a SFR between $s := \Psi \circ \Phi u$ and $s' := \Psi \circ \Phi u'$

Step 3: Connectivity

There must be at least two non adjacent edges incident to f in G^* not included in s' separating a and b. Call these missing edges e^*_+ and e^*_-. In G they correspond to edges connecting v to the vertices w_+ and w_- respectively. They must have been using the same strategy in u.

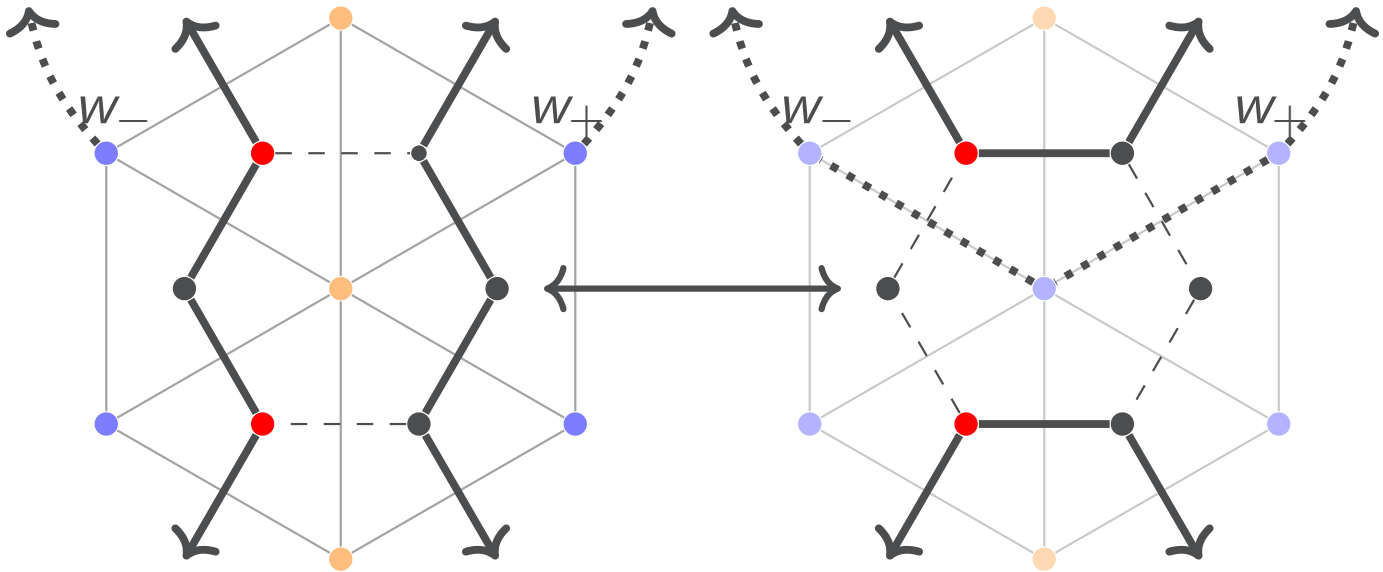


Proof of the Quasiisometry

Assume that $d(u, u') = 1$ and that $\Phi u \in \mathcal{QC}_G$. We will show that there is a SFR between $s := \Psi \circ \Phi u$ and $s' := \Psi \circ \Phi u'$

Step 3: Connectivity

There must be at least two non adjacent edges incident to f in G^* not included in s' separating a and b. Call these missing edges e^*_+ and e^*_-. In G they correspond to edges connecting v to the vertices w^+ and w_- respectively. They must have been using the same strategy in u.

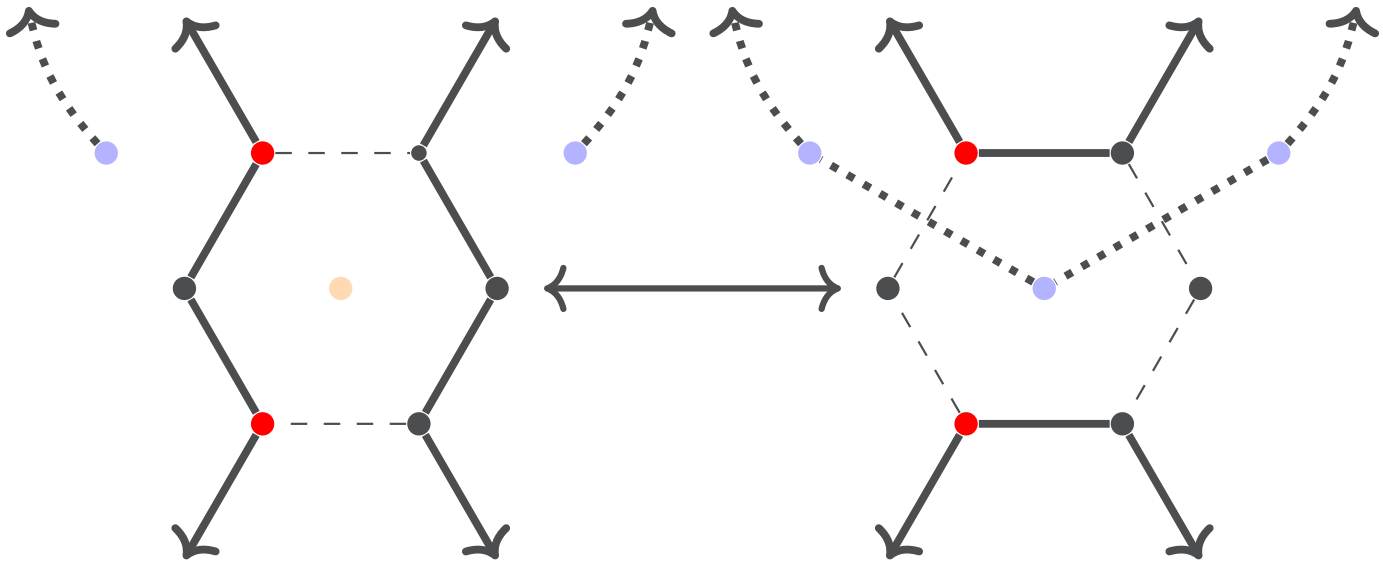


Proof of the Quasiisometry

Assume that $d(u, u') = 1$ and that $\Phi u \in \mathcal{QC}_G$. We will show that there is a SFR between $s := \Psi \circ \Phi u$ and $s' := \Psi \circ \Phi u'$

Step 3: Connectivity

Therefore, there is a cycle starting and ending at v of players all using the same strategy in u'. This cycle defines a Jordan curve in the plane which separates a from b no edge of s' can cross the Jordan curve by construction so a and b are disconnected



Equilibrium Result

Lemma 3.6

For a planar 3-edge connected graph G, if $u \in \mathcal{A}_G$ has $\Psi u \in \mathcal{QC}_G$ and $d(u, u') = 1$ in A for some u', then $D(\Psi \circ \Phi u, \Psi \circ \Phi u') = 1$ in $\mathcal{SC}_{G^*}$

Equilibrium Result

Lemma 3.6

For a planar 3-edge connected graph G, if $u \in \mathcal{A}_G$ has $\Psi u \in \mathcal{QC}_G$ and $d(u, u') = 1$ in A for some u', then $D(\Psi \circ \Phi u, \Psi \circ \Phi u') = 1$ in $\mathcal{SC}_{G^*}$

Theorem 3.2

If $s \in \mathcal{SC}_{G^*}$ satisfies

$$size(s) \leq size(r) \quad \forall r \in \Gamma_{G^*}(s)$$

where $\Gamma_{G^*}(s) := \{r \in \mathcal{SC}_{G^*}; D(s, r) \leq 1\}$ then there is a $u \in \Phi^{-1} \circ \Psi^{-1} s$ so that u is a Nash equilibrium

Equilibrium Result

Lemma 3.6

For a planar 3-edge connected graph G, if $u \in \mathcal{A}_G$ has $\Psi u \in \mathcal{QC}_G$ and $d(u, u') = 1$ in A for some u', then $D(\Psi \circ \Phi u, \Psi \circ \Phi u') = 1$ in $\mathcal{SC}_{G^*}$

Theorem 3.2

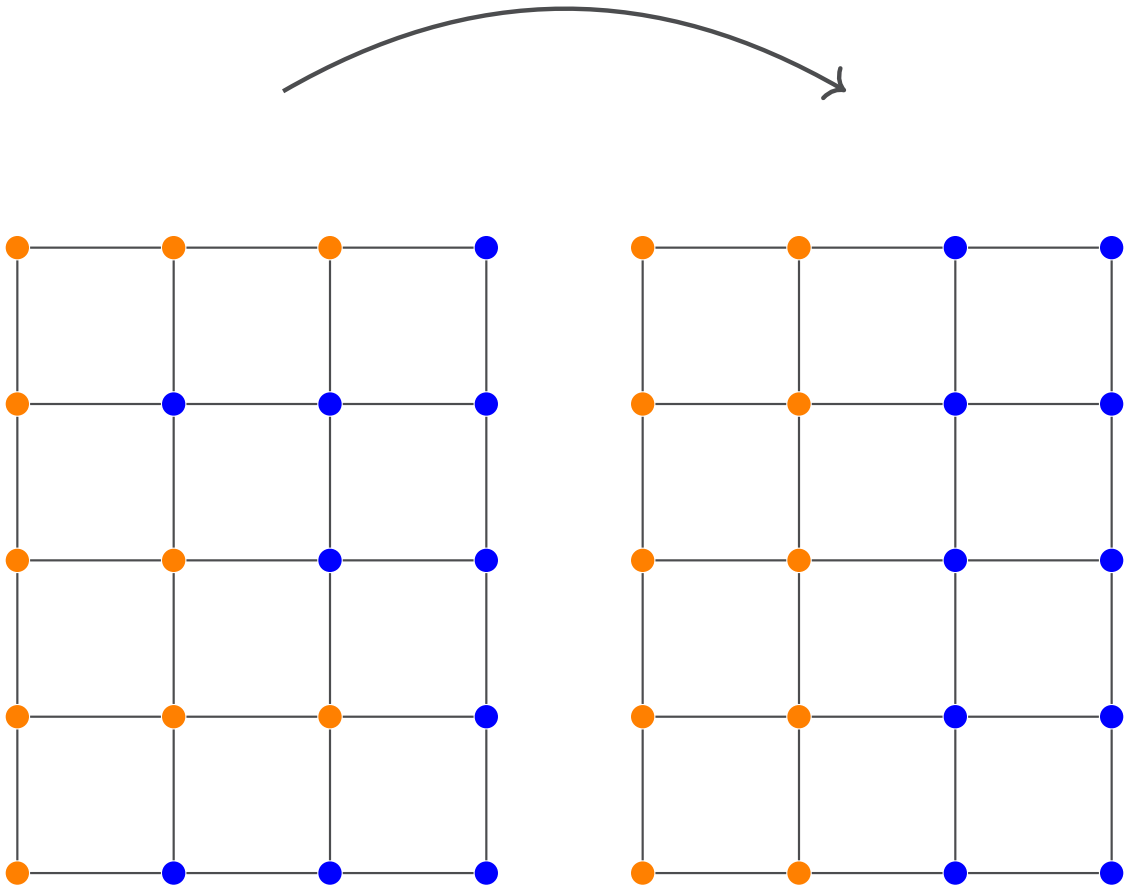
If $s \in \mathcal{SC}_{G^*}$ satisfies

$$size(s) \leq size(r) \quad \forall r \in \Gamma_{G^*}(s)$$

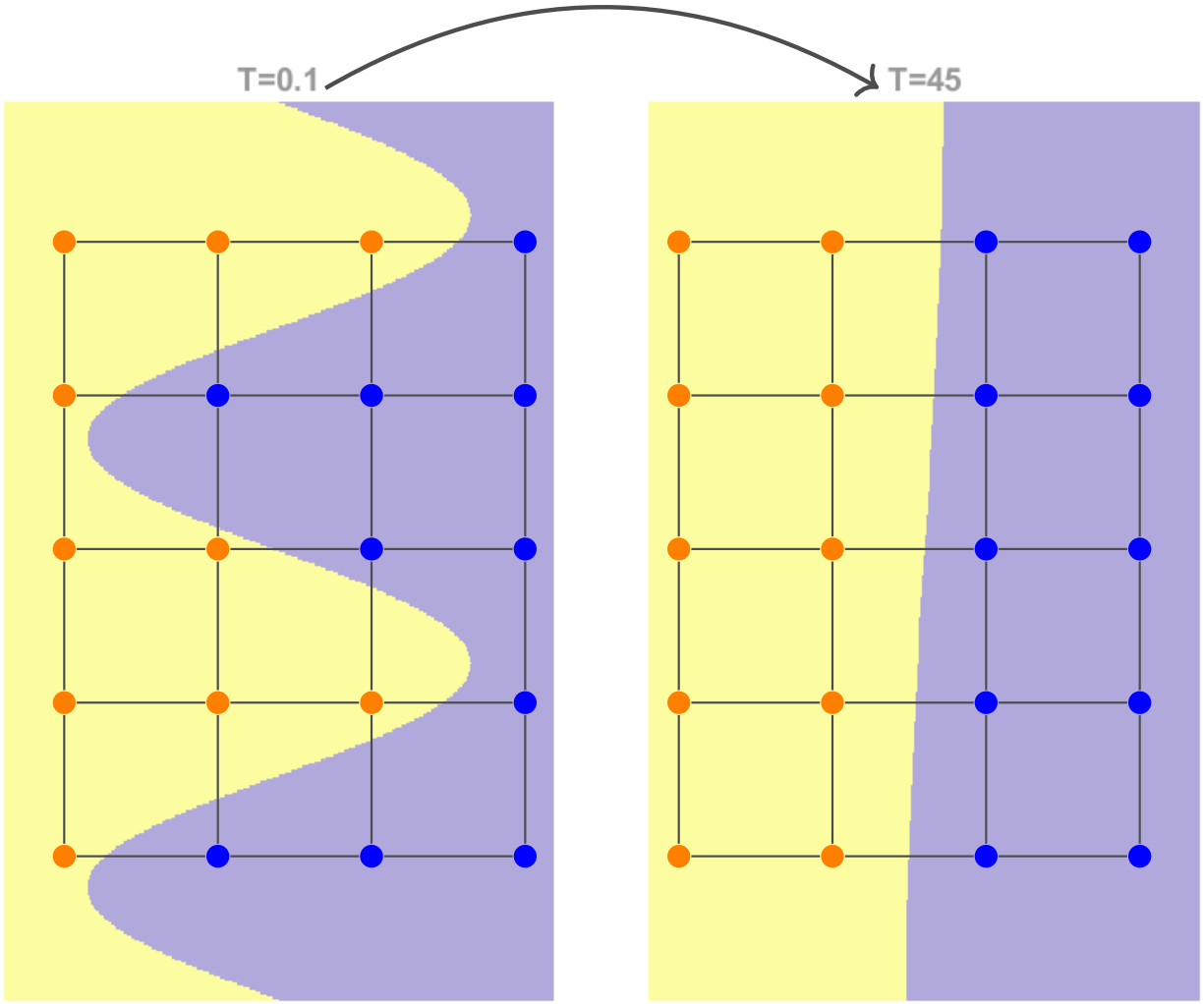
where $\Gamma_{G^*}(s) := \{r \in \mathcal{SC}_{G^*}; D(s, r) \leq 1\}$ then there is a $u \in \Phi^{-1} \circ \Psi^{-1} s$ so that u is a Nash equilibrium

- When we restrict to two strategies MBR is equivalent to a "cycle shortening flow" (Thm 3.5)
- In a continuous player space MBR recapitulates mean curvature flow in the Cauchy setting (Thm 3.6)

Boundary Minimization



Boundary Minimization



Results of the Dual Model

	Simulation	Dual Model
Main Achievement	Observation	
Numerical Speed	✓	
Analytical Tractability	X	
Model Accuracy	✓ ✓	

Results of the Dual Model

	Simulation	Dual Model
Main Achievement	Observation	Intuition
Numerical Speed	✓	
Analytical Tractability	X	
Model Accuracy	✓ ✓	

Results of the Dual Model

	Simulation	Dual Model
Main Achievement	Observation	Intuition
Numerical Speed	✓	X
Analytical Tractability	X	
Model Accuracy	✓ ✓	

Results of the Dual Model

	Simulation	Dual Model
Main Achievement	Observation	Intuition
Numerical Speed	✓	X
Analytical Tractability	X	✓
Model Accuracy	✓ ✓	

Results of the Dual Model

	Simulation	Dual Model
Main Achievement	Observation	Intuition
Numerical Speed	✓	X
Analytical Tractability	X	✓
Model Accuracy	✓ ✓	✓

Results of the Dual Model

	Simulation	Dual Model
Main Achievement	Observation	Intuition
Numerical Speed	✓	X
Analytical Tractability	X	✓
Model Accuracy	✓ ✓	✓ ✓

Results of the Dual Model

	Simulation	Dual Model
Main Achievement	Observation	Intuition
Numerical Speed	✓	X
Analytical Tractability	X	✓
Model Accuracy	✓ ✓	✓ ✓

Main Result

The strategic boundaries minimize their lengths (area) in a parabolic, diffusive manner. Large regions of consensus are unlikely to change.

Replicator Dynamics

Replicator Equation

$$\frac{d}{dt}p_i = p_i(f_i(p) - \varphi(p))$$

Replicator Dynamics

Replicator Equation

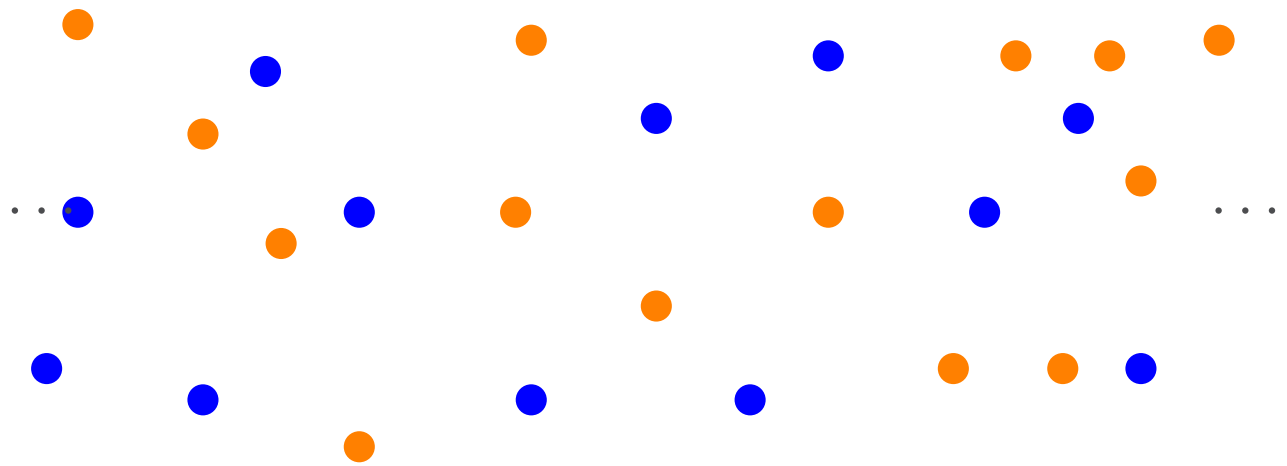
$$\frac{d}{dt}p_i = p_i(f_i(p) - \varphi(p))$$

In an infinite well mixed population p is a distribution over pure strategies $\{p_1, ..., p_m\}$, $f_i(p)$ is the fitness of playing i against p and $\varphi(p)$ is the average fitness of a player in p.

Replicator Dynamics

Replicator Equation

$$\frac{d}{dt}p_i = p_i(f_i(p) - \varphi(p))$$



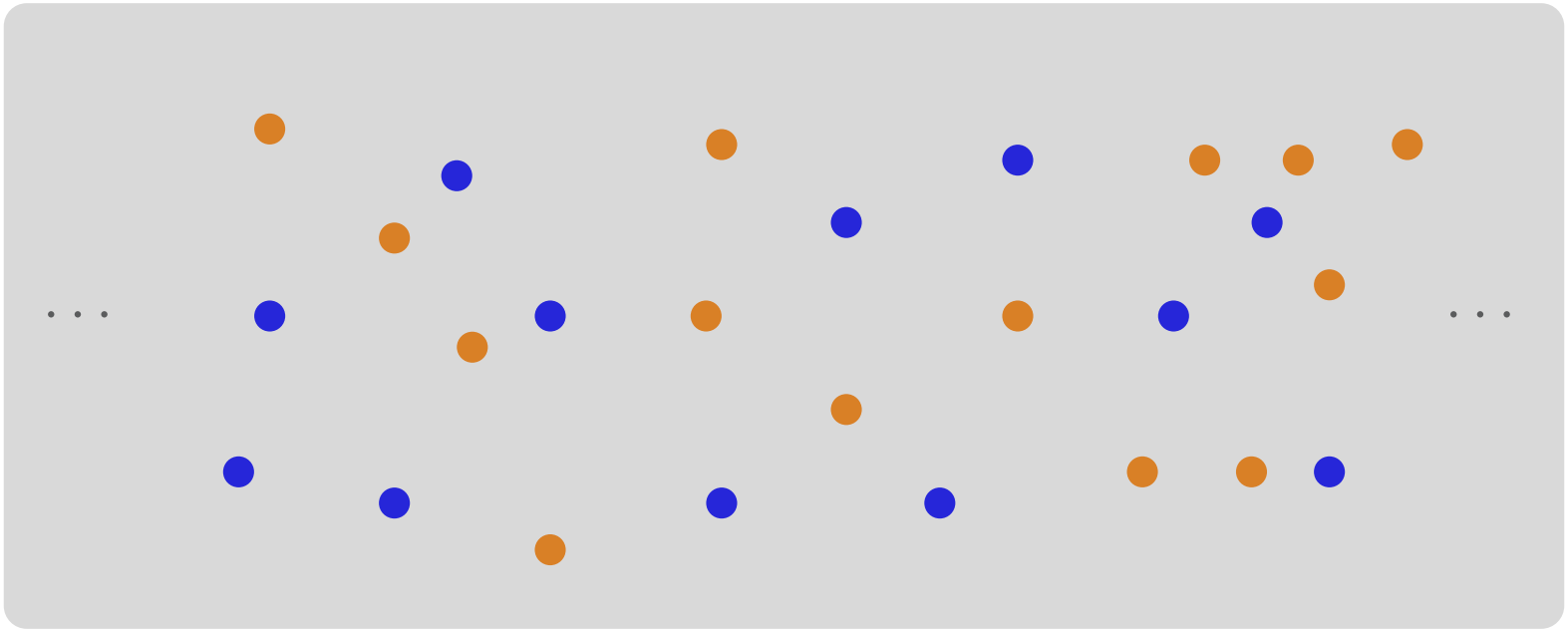
Original Setting

$$V = \mathbb{N} \qquad C = \{[1,0],[0,1]\} \subset \mathbb{R}^m \qquad p \in \Delta^{m-1}$$

Replicator Dynamics

Replicator Equation

$$\frac{d}{dt}p_i = p_i(f_i(p) - \varphi(p))$$



Modifying the model to add structure

Consider this "infinite population" as a single patch

Replicator Dynamics

Replicator Equation

$$\frac{d}{dt}p_i = p_i(f_i(p) - \varphi(p))$$

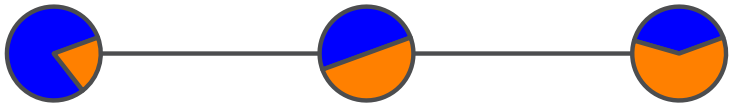
Modifying the model to add structure

Now for a single player, $u_v \in C := \Delta^{m-1}$

Replicator Dynamics

Replicator Equation

$$\frac{d}{dt}p_i = p_i(f_i(p) - \varphi(p))$$



Modifying the model to add structure

$$f_v^i(u) = \sum_{w \in V} W_{w,v}\langle \hat{e}^i, Au_w \rangle \qquad \varphi_v(u) = \sum_{w \in V} W_{w,v}\langle u_v, Au_w \rangle$$

Structured Replicator Dynamics

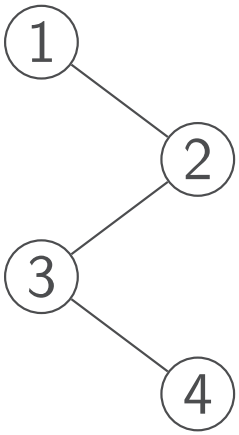
A strategy profile is $u : V \to \Delta^{m-1}$ which is identical to $u \in (\Delta^{m-1})^V$. Thus we have an $n \cdot m$ dimensional system of ODEs

ODE System

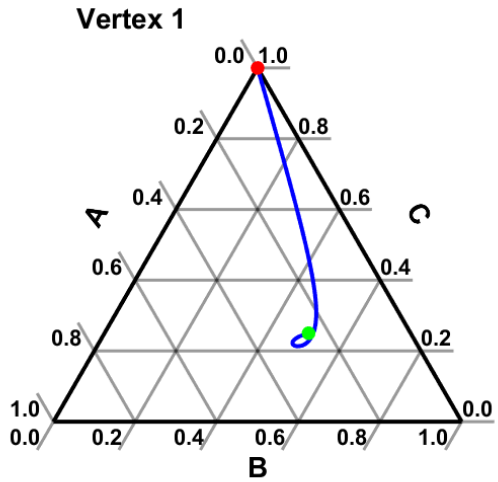
$$\frac{d}{dt} u_v^i = u_v^i \langle \hat{e}^i - u_v, \sum_{w \in V} A u_v \rangle$$

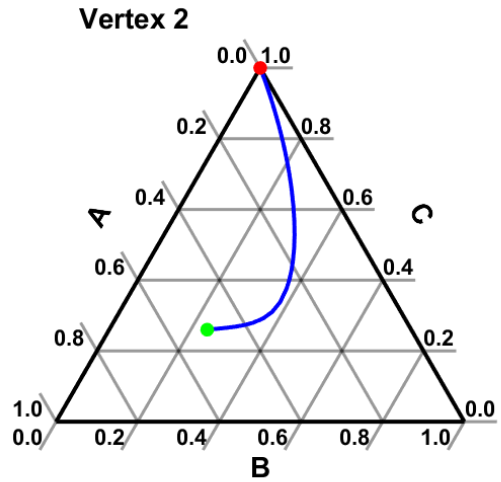
In general, lower indices indicate players and upper indices indicate strategies.

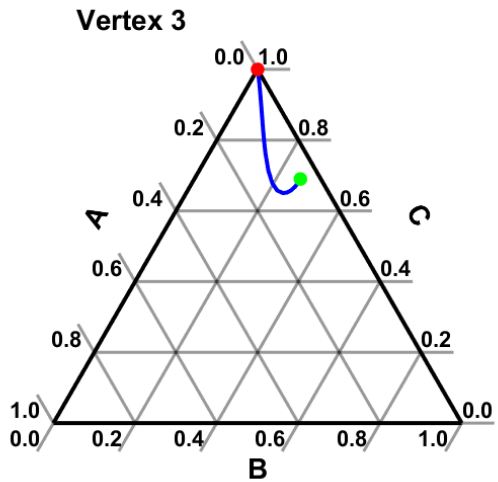
Examples

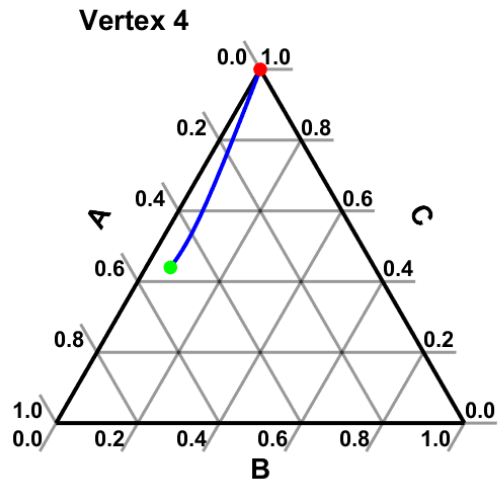


$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

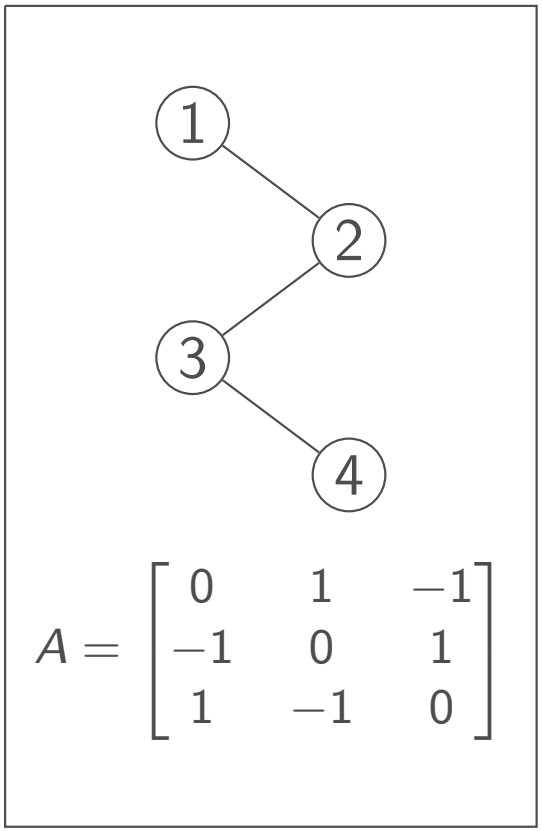




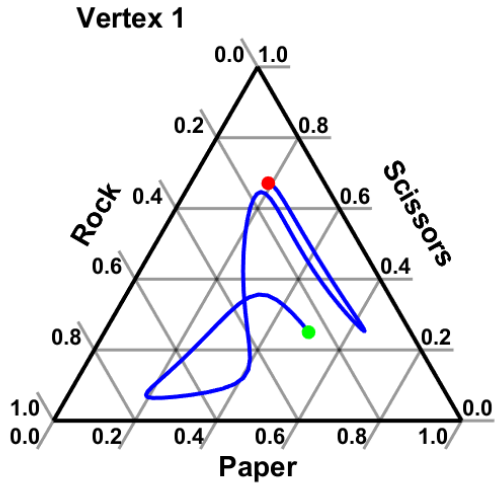


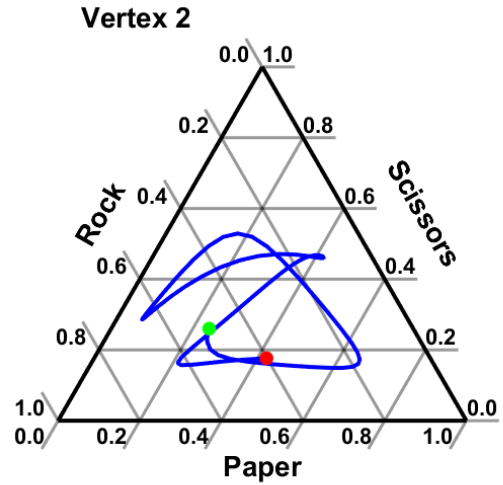


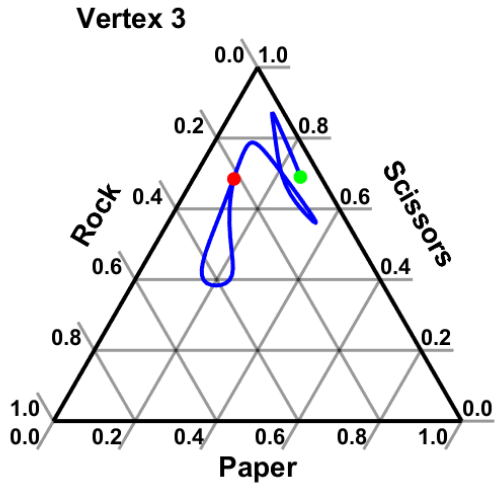
Examples

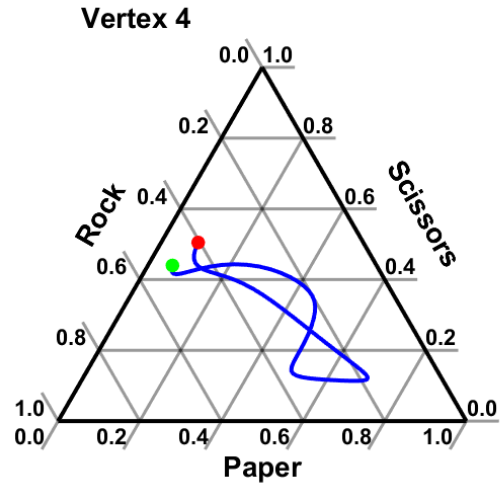


$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$









Dynamic results

How do strategy profiles change in this model compared to the discrete time model?

Dynamic results

How do strategy profiles change in this model compared to the discrete time model?

Lemma 4.3

If $u(t)$ solves the IVP with $u(0) \in (\Delta^{m-1})^V$ then

$$\left.\frac{d}{dt}\right|_{\zeta=0} w_v(u_v + \zeta \frac{d}{dt} u_v | u) \geq 0$$

with equality only when $\frac{d}{dt} u_v = \vec{0}$. This means the replicator dynamics represent a *Better Response* strategy revision protocol.

Better Response

The fact that this is a "better response" strategy revision protocol means

- transient behavior of the ODE system may not exactly reflect the behavior of the discrete time game
- basins of stability are not necessarily the same across the two models

However, we can examine equilibria and their stability.

Theorem 4.1

If $u(t)$ is a convergent trajectory in the interior of $(\Delta^{m-1})^V$ then $\lim_{t\to\infty} u(t)$ exists and is a Nash equilibrium.

Persistence of Mixed strategies

Stability results from ODE theory allow us to understand more about coordinating systems. for instance:

> *A strategy profile where adjacent players are playing overlapping mixed strategies is always unstable.*

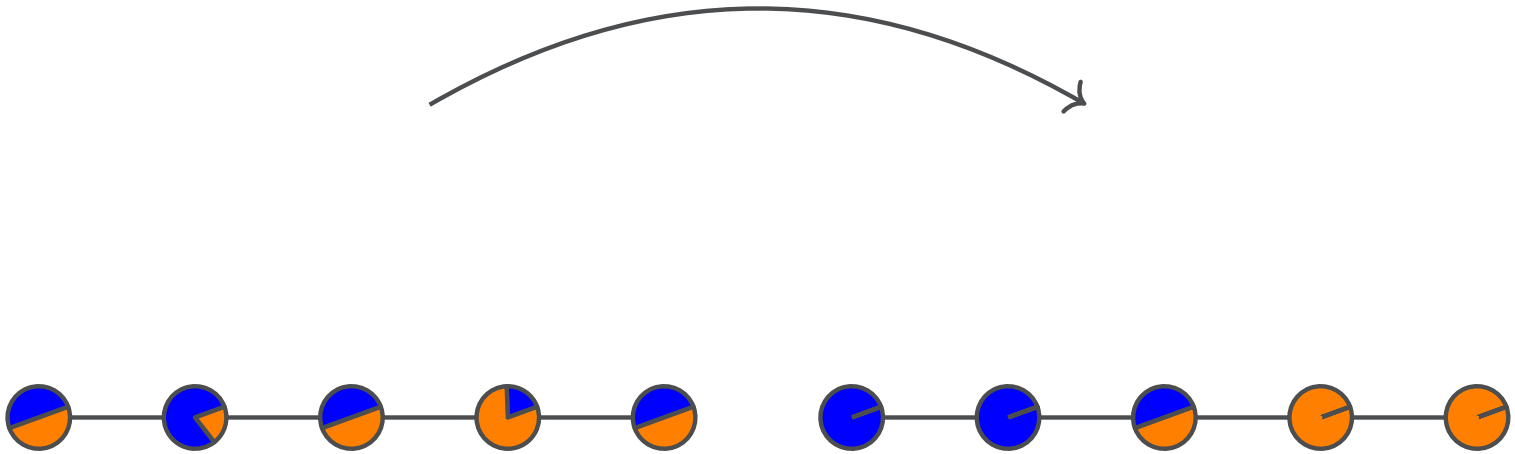
Lemma 4.9

In a pure coordination game, if an equilibrium strategy profile u^* has two players v and w satisfying

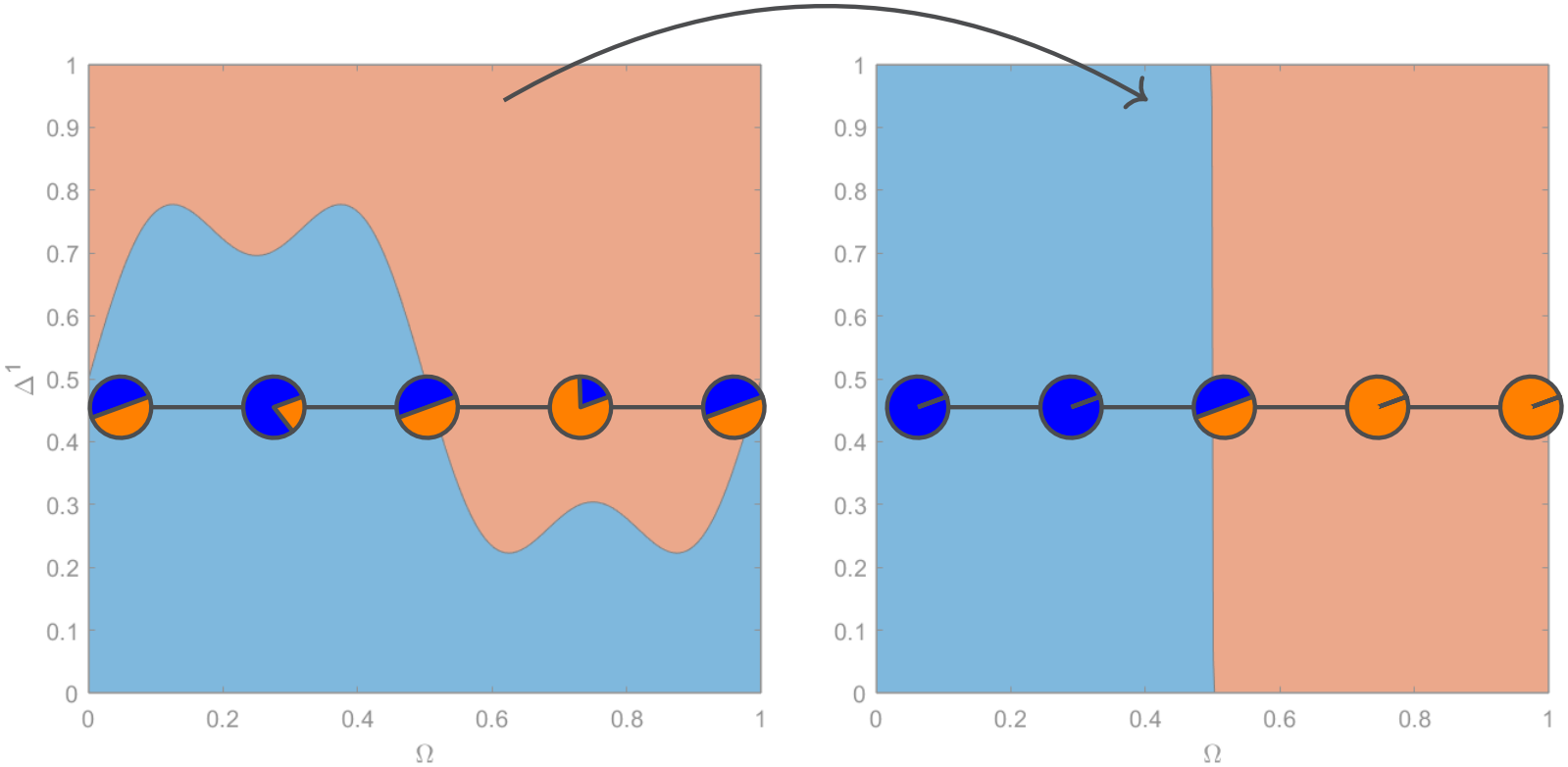
> There exists $i \in BR_v(u^*) \cap BR_w(u^*)$ such that either $u_v^i > 0$ and $u_w^i > 0$ or $u_v^i = 0$ and $u_w^i < 1$
>
> $|BR_v(u^*)| > 1$ and $|BR_w(u^*)| > 1$

then u^* is unstable.

Discretization



Discretization



Results of the ODE Model

	Simulation	Dual Model	ODE Model
Main Achievement	Observation	Intuition	
Numerical Speed	✓	X	
Analytical Tractability	X	✓	
Model Accuracy	✓ ✓	✓ ✓	

Results of the ODE Model

	Simulation	Dual Model	ODE Model
Main Achievement	Observation	Intuition	Stability
Numerical Speed	✓	X	
Analytical Tractability	X	✓	
Model Accuracy	✓ ✓	✓ ✓	

Results of the ODE Model

	Simulation	Dual Model	ODE Model
Main Achievement	Observation	Intuition	Stability
Numerical Speed	✓	X	✓
Analytical Tractability	X	✓	
Model Accuracy	✓ ✓	✓ ✓	

Results of the ODE Model

	Simulation	Dual Model	ODE Model
Main Achievement	Observation	Intuition	Stability
Numerical Speed	✓	X	✓
Analytical Tractability	X	✓	✓
Model Accuracy	✓ ✓	✓ ✓	

Results of the ODE Model

	Simulation	Dual Model	ODE Model
Main Achievement	Observation	Intuition	Stability
Numerical Speed	✓	X	✓
Analytical Tractability	X	✓	✓
Model Accuracy	✓ ✓	✓ ✓	X

Results of the ODE Model

	Simulation	Dual Model	ODE Model
Main Achievement	Observation	Intuition	Stability
Numerical Speed	✓	X	✓
Analytical Tractability	X	✓	✓
Model Accuracy	✓ ✓	✓ ✓	X ✓

Results of the ODE Model

	Simulation	Dual Model	ODE Model
Main Achievement	Observation	Intuition	Stability
Numerical Speed	✓	X	✓
Analytical Tractability	X	✓	✓
Model Accuracy	✓ ✓	✓ ✓	X ✓

Main Result

Strategic gradients are unstable. Regions with high strategic gradients separate and regions with low strategic gradients, near pure strategies, diffuse.

Nonlocal diffusion

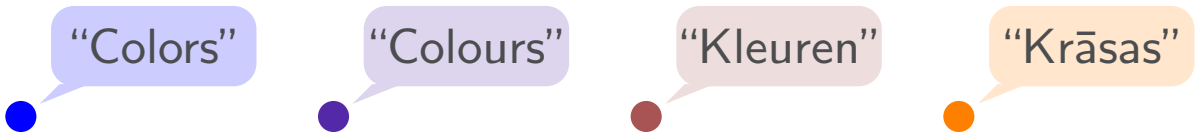
Comparable Strategies

What if two strategies, which are not identical, can still offer a positive payoff by being considered "close enough"

Nonlocal diffusion

Comparable Strategies

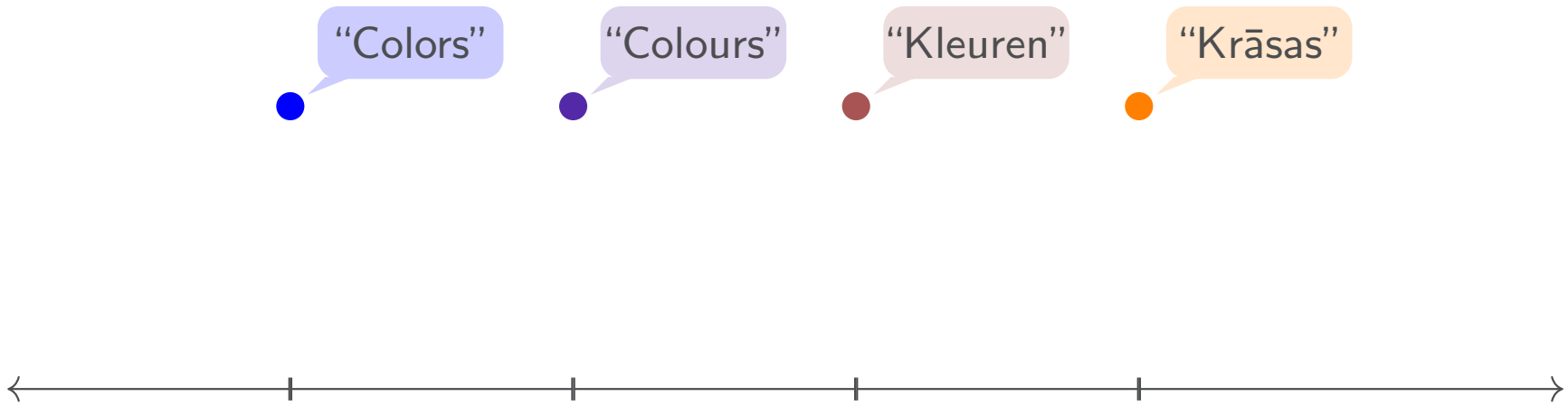
What if two strategies, which are not identical, can still offer a positive payoff by being considered "close enough"



Nonlocal diffusion

Comparable Strategies

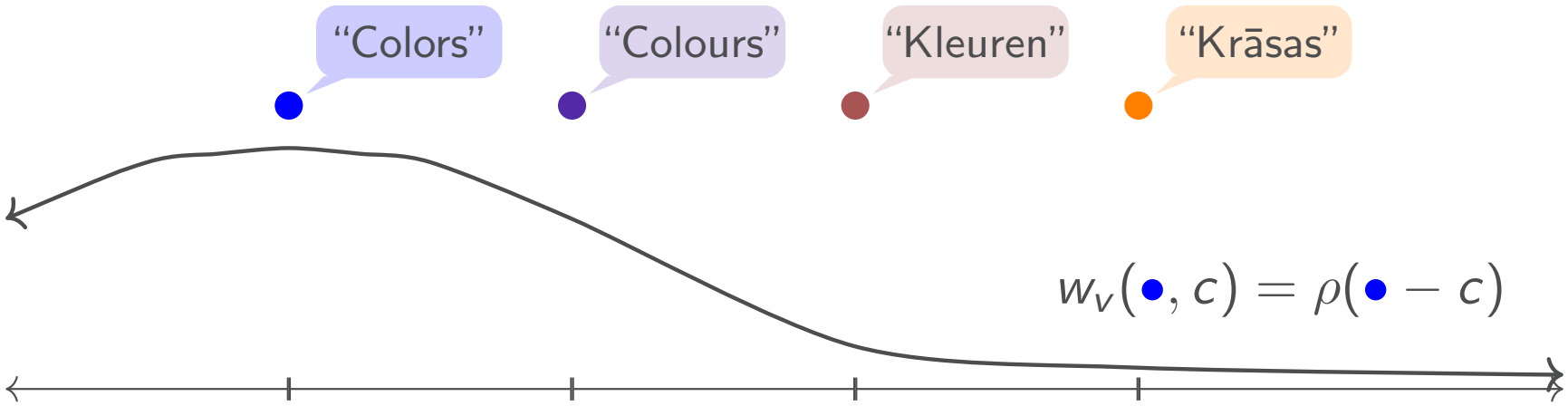
What if two strategies, which are not identical, can still offer a positive payoff by being considered "close enough"



Nonlocal diffusion

Comparable Strategies

What if two strategies, which are not identical, can still offer a positive payoff by being considered "close enough"



Continuous Strategy Space

Discrete strategy Space

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \rightarrow \quad A' = \begin{bmatrix} 0.9 & 0.05 & 0 \\ 0.05 & 0.9 & 0.05 \\ 0 & 0.05 & 0.9 \end{bmatrix}$$

In this case, the payoff from playing pure strategies can be determined simply be the "distance" between those strategies

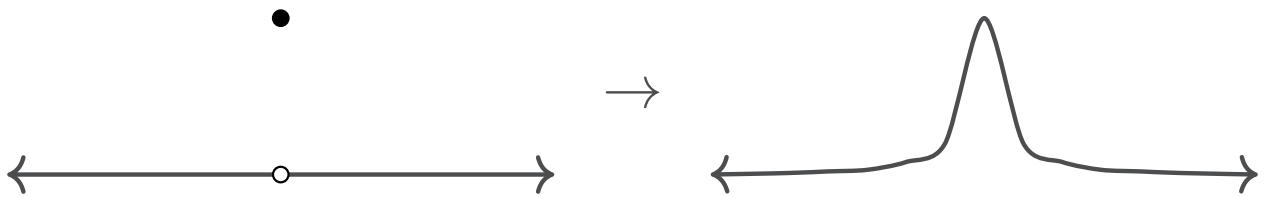
Continuous Strategy Space

Discrete strategy Space

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \rightarrow \quad A' = \begin{bmatrix} 0.9 & 0.05 & 0 \\ 0.05 & 0.9 & 0.05 \\ 0 & 0.05 & 0.9 \end{bmatrix}$$

In this case, the payoff from playing pure strategies can be determined simply be the "distance" between those strategies

Continuous Strategy Space

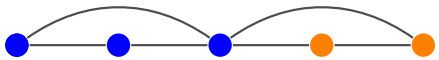


$\rho(z)$ is the payoff from playing a strategy c against strategy c' so that $c - c' = z$

Continuous Player Space

Discrete Player Space

Players V are vertices in the graph G



v

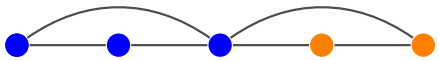
Payoffs are computed as a sum over all vertices weighted by $W_{v,w}$

$$\sum_{w \in V} W_{w,v} \rho(u_v - u_w)$$

Continuous Player Space

Discrete Player Space

Players V are vertices in the graph G



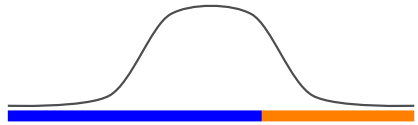
v

Payoffs are computed as a sum over all vertices weighted by $W_{v,w}$

$$\sum_{w \in V} W_{w,v} \rho(u_v - u_w)$$

Continuous Player Space

Players x are points in $\Omega \subset \mathbb{R}^n$



x

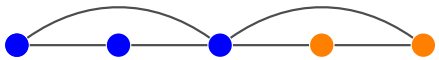
Payoffs are computed as in integral over Ω weighted by $K(x, y)$.

$$\int_{\Omega} K(x, y) \rho(u(x) - u(y)) dy$$

Continuous Player Space

Discrete Player Space

Players V are vertices in the graph G



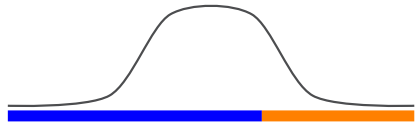
V

Payoffs are computed as a sum over all vertices weighted by $W_{v,w}$

$$\sum_{w \in V} W_{w,v} \rho(u_v - u_w)$$

Continuous Player Space

Players x are points in $\Omega \subset \mathbb{R}^n$



x

Payoffs are computed as in integral over Ω weighted by $K(x, y)$.

$$\int_{\Omega} K(x, y) \rho(u(x) - u(y)) dy$$

$$u \in W^{1,\infty}(\Omega)$$

Strategy Revision Protocol

We will be using the Myopic Best Response strategy revision protocol, but with modification

Propositon 5.1

Under myopic best response, bounded strategy profiles of the game with players $\Omega \subseteq \mathbb{R}^n$ choosing strategies in $\mathbb{R}$ will evolve as

$$\frac{\partial}{\partial t}u(x,t) = g[u](x,t) := \int_\Omega K(x,y)\rho'(u(x,t)-u(y,t))dy$$

so long as the following three hypotheses are met

(H1) Players change their strategies in arbitrarily small time steps Δt

(H2) Players incur a quadratic cost $\frac{h^2}{\Delta t}$ for changing their strategy

(H3) $\rho \in C^{1,1}(\mathbb{R})$ and $\rho(z) \leq Cz^2 + A$ for some nonnegative C, A

Existence, Uniqueness, Regularity

Theorem 5.1

The initial value problem $u_t = g[u]$ in Ω_τ has a unique continuous and bounded solution in Ω_τ for some time τ when $u(x,0) = u_0 \in C_b^0(\Omega)$

Existence, Uniqueness, Regularity

Theorem 5.1

The initial value problem $u_t = g[u]$ in Ω_τ has a unique continuous and bounded solution in Ω_τ for some time τ when $u(x,0) = u_0 \in C_b^0(\Omega)$

Theorem 5.2

If ρ satisfies $z \cdot \rho'(z) \leq 0$ (The coordination condition), then the initial value problem $u_t = g[u]$ in $\Omega \times \mathbb{R}_+$ has a unique continuous and bounded solution in $\Omega \times \mathbb{R}_+$ when $u(x,0) = u_0 \in C_b^0(\Omega)$.

Existence, Uniqueness, Regularity

Theorem 5.1

The initial value problem $u_t = g[u]$ in Ω_τ has a unique continuous and bounded solution in Ω_τ for some time τ when $u(x,0) = u_0 \in C_b^0(\Omega)$

Theorem 5.2

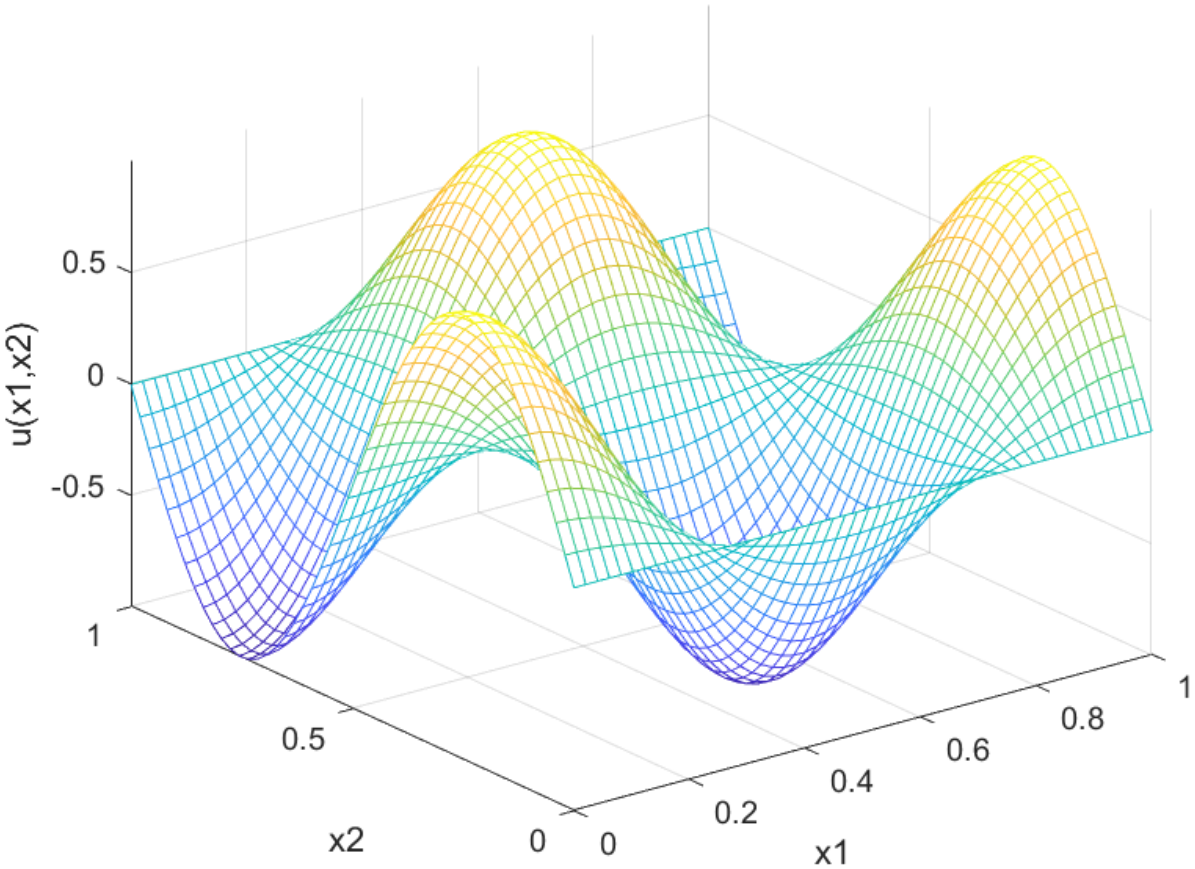
If ρ satisfies $z \cdot \rho'(z) \leq 0$ (The coordination condition), then the initial value problem $u_t = g[u]$ in $\Omega \times \mathbb{R}_+$ has a unique continuous and bounded solution in $\Omega \times \mathbb{R}_+$ when $u(x,0) = u_0 \in C_b^0(\Omega)$.

Theorem 5.3

If ρ satisfies the coordination condition and $u_t = g[u]$ in Ω_T and $u_0 \in C_b^{0,1}(\Omega)$ then $u \in W^{1,\infty}(\Omega_T)$ with

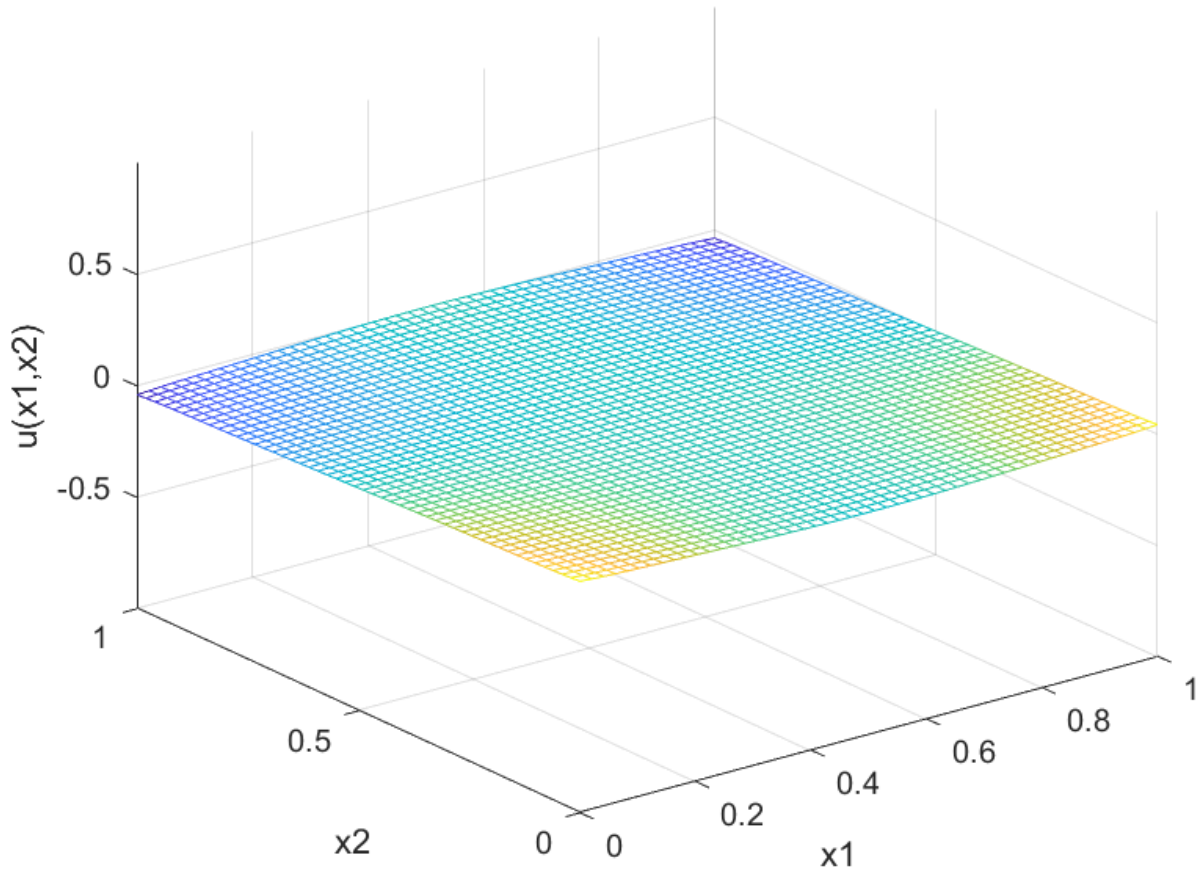
$$\|Du\|_{L^\infty(\Omega)} \leq (L_0 + CT)e^{cT}$$

Numerical Simulations



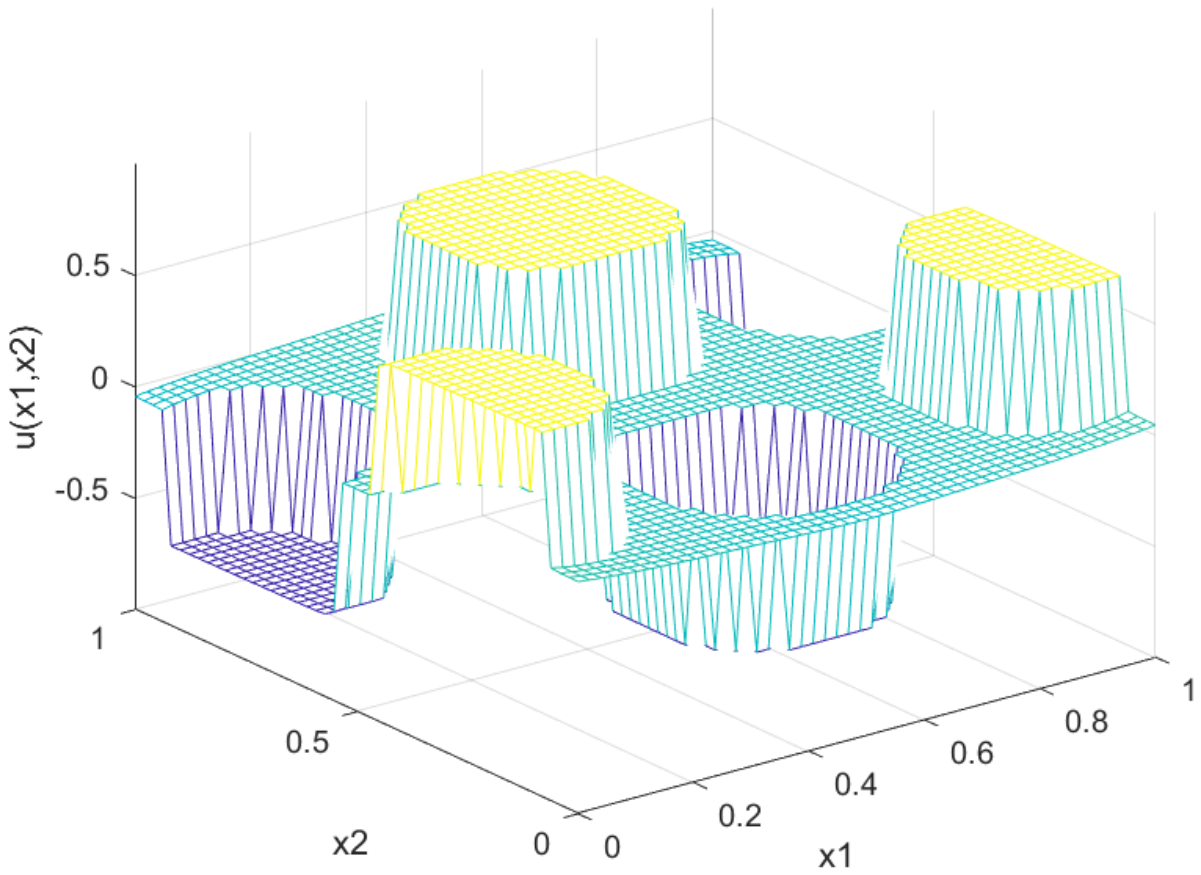
The nonlocal diffusion equation is highly sensitive to changes in the recognition function

Numerical Simulations



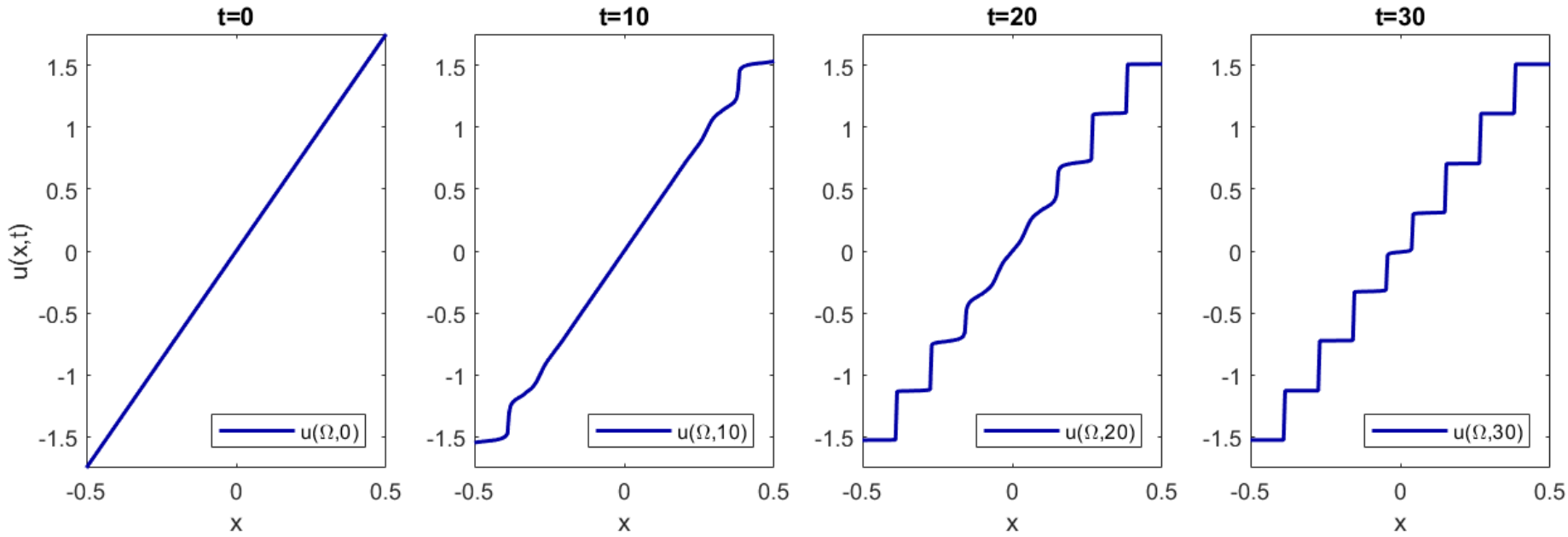
The nonlocal diffusion equation is highly sensitive to changes in the recognition function

Numerical Simulations



The nonlocal diffusion equation is highly sensitive to changes in the recognition function

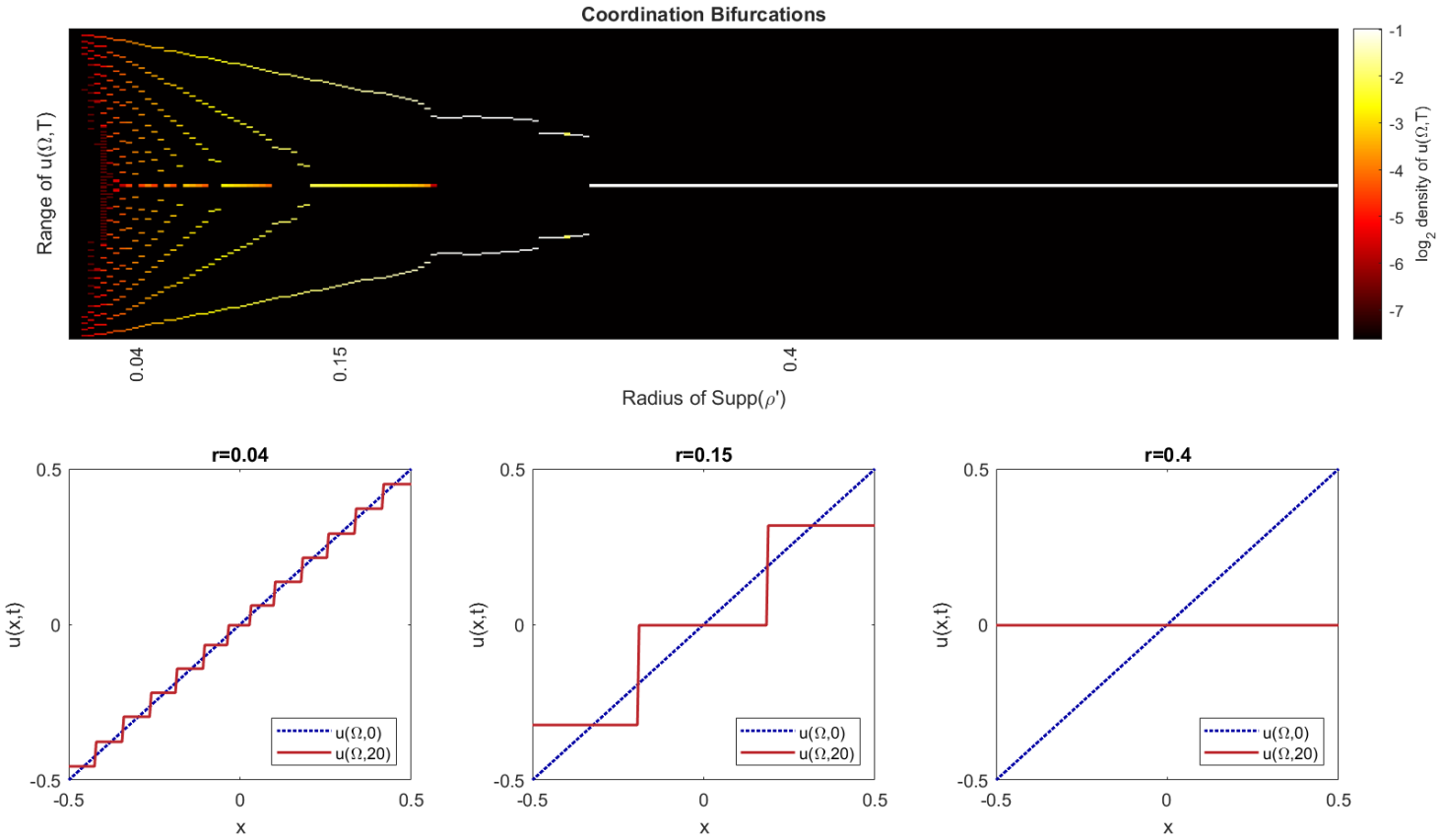
Discretization



Whenever $supp(\rho') \subset [-a, a]$ with only one zero in the interior of its support, then we always observe discretization

Discretization

The gaps between the points in the image of the limit are spaced in a predicable way which is related to the recognition function.



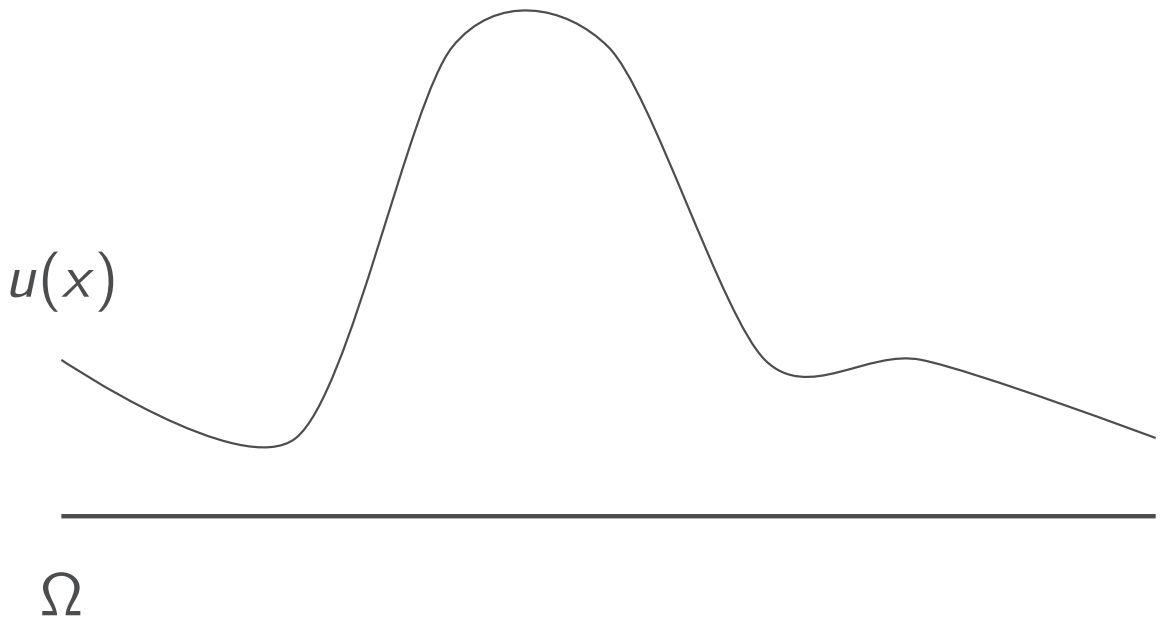
Discretization

The gaps between the points in the image of the limit are spaced in a predicable way which is related to the recognition function.

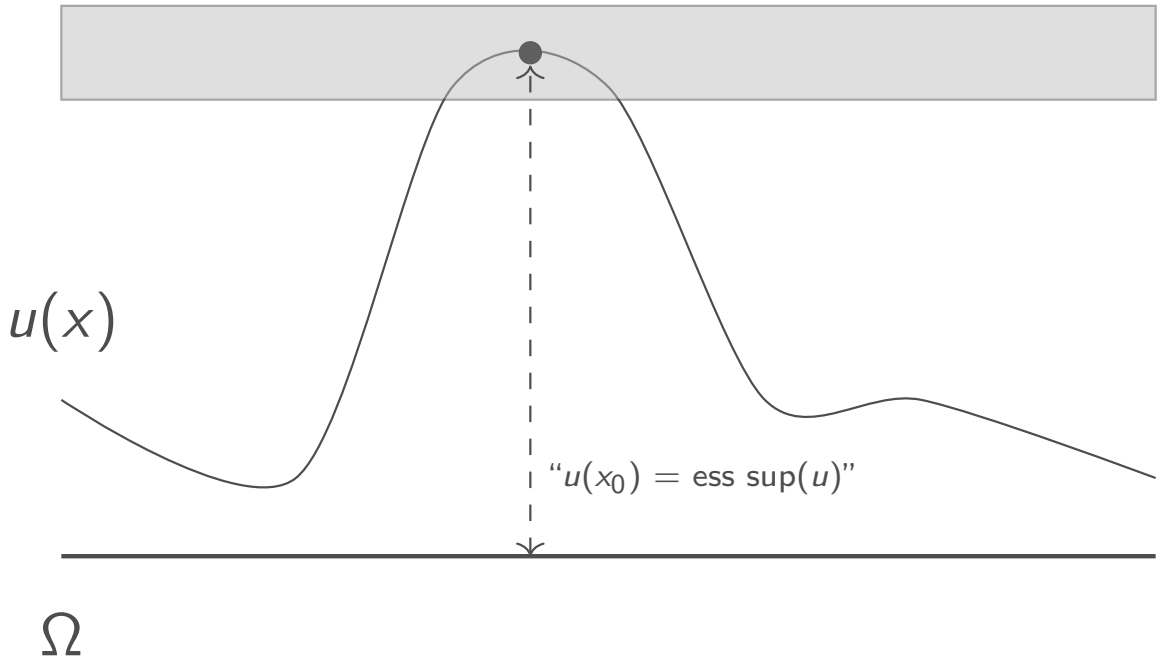
Theorem 5.6

Let Ω be a bounded domain in $\mathbb{R}^n$ and $K(x,y) \in C_b^0(\Omega; L^1(\Omega))$ be bounded below by λ, so that $\Omega \subset supp(K(x,\cdot))$. Let ρ' satisfy the coordination condition and have support $(-a,a) \subset \mathbb{R}$ with only one zero on the interior of its support. Under these conditions, if u satisfies $g[u] = 0$ in Ω and u is bounded then the image $u(\Omega)$ is a finite set of points separated by at least $|a|$ except possible at a set of measure 0.

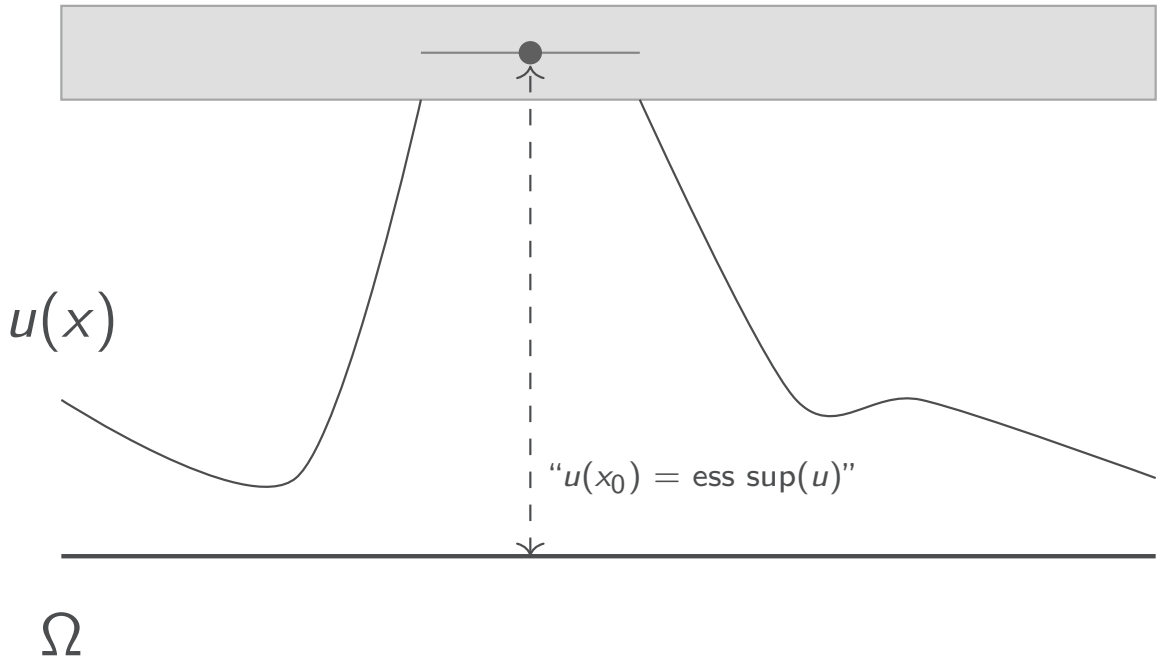
Proof Sketch



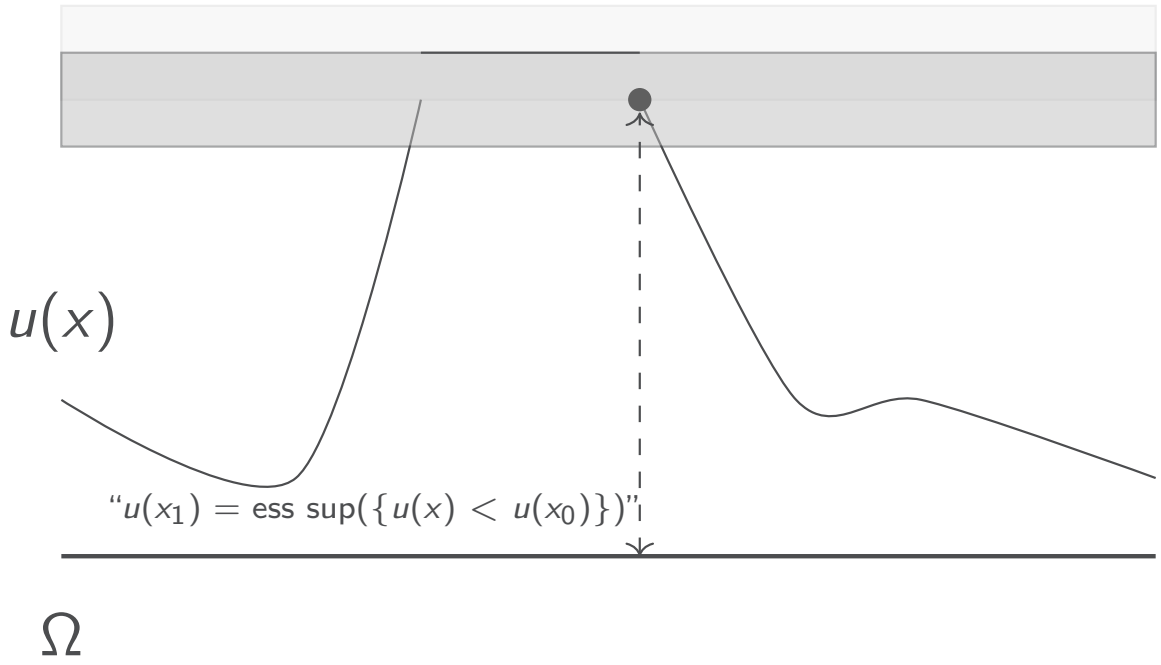
Proof Sketch



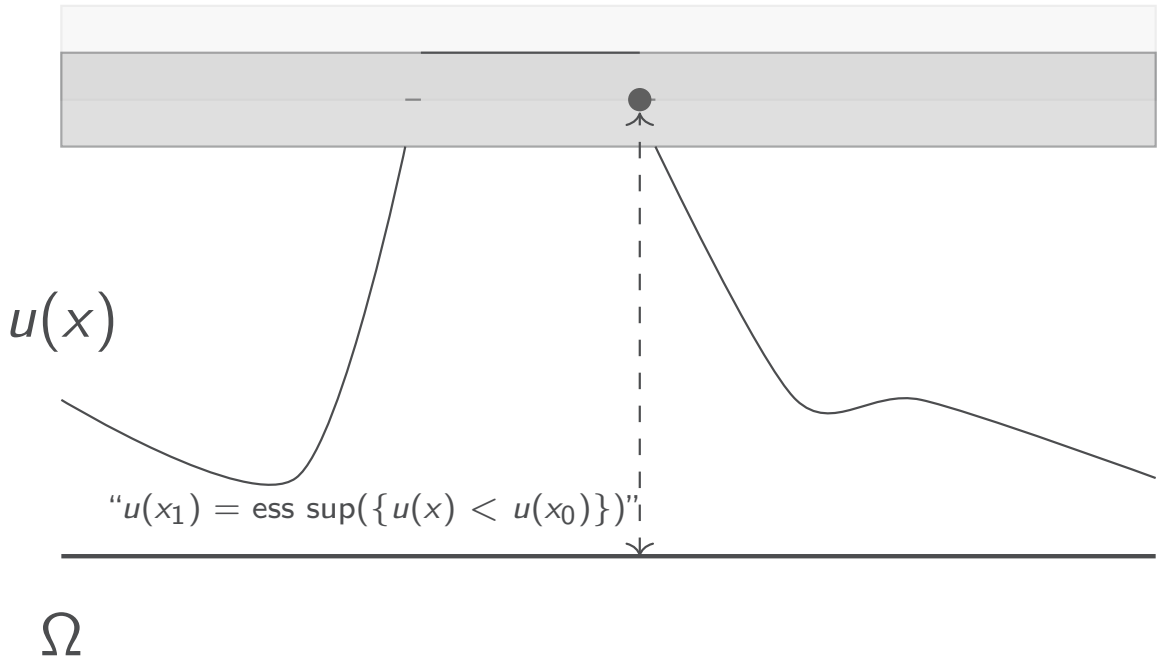
Proof Sketch



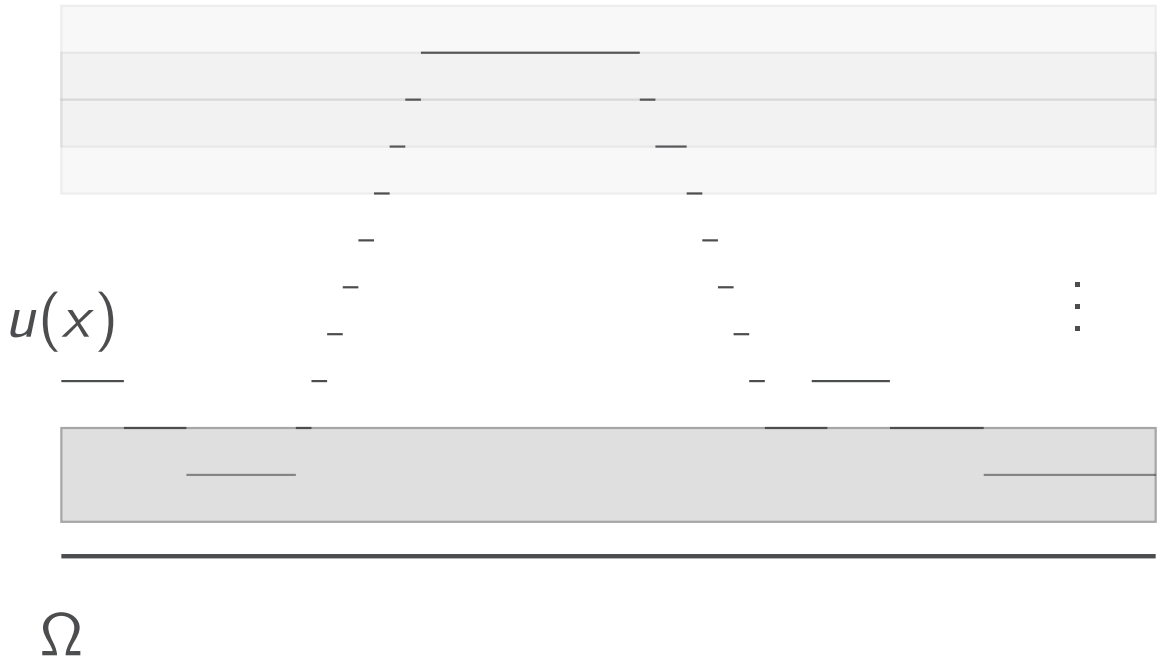
Proof Sketch



Proof Sketch



Proof Sketch



Results of the nonlocal model

	Simulation	Dual Model	ODE Model	Nonlocal Model
Main Achievement	Observation	Intuition	Stability	
Numerical Speed	✓	X	✓	
Analytical Tractability	X	✓	✓	
Model Accuracy	✓ ✓	✓ ✓	X ✓	

Results of the nonlocal model

	Simulation	Dual Model	ODE Model	Nonlocal Model
Main Achievement	Observation	Intuition	Stability	Extension
Numerical Speed	✓	X	✓	
Analytical Tractability	X	✓	✓	
Model Accuracy	✓ ✓	✓ ✓	X ✓	

Results of the nonlocal model

	Simulation	Dual Model	ODE Model	Nonlocal Model
Main Achievement	Observation	Intuition	Stability	Extension
Numerical Speed	✓	X	✓	✓
Analytical Tractability	X	✓	✓	
Model Accuracy	✓ ✓	✓ ✓	X ✓	

Results of the nonlocal model

	Simulation	Dual Model	ODE Model	Nonlocal Model
Main Achievement	Observation	Intuition	Stability	Extension
Numerical Speed	✓	X	✓	✓
Analytical Tractability	X	✓	✓	✓
Model Accuracy	✓ ✓	✓ ✓	X ✓	

Results of the nonlocal model

	Simulation	Dual Model	ODE Model	Nonlocal Model
Main Achievement	Observation	Intuition	Stability	Extension
Numerical Speed	✓	X	✓	✓
Analytical Tractability	X	✓	✓	✓
Model Accuracy	✓ ✓	✓ ✓	X ✓	✓

Results of the nonlocal model

	Simulation	Dual Model	ODE Model	Nonlocal Model
Main Achievement	Observation	Intuition	Stability	Extension
Numerical Speed	✓	X	✓	✓
Analytical Tractability	X	✓	✓	✓
Model Accuracy	✓ ✓	✓ ✓	X ✓	✓ X

Results of the nonlocal model

	Simulation	Dual Model	ODE Model	Nonlocal Model
Main Achievement	Observation	Intuition	Stability	Extension
Numerical Speed	✓	X	✓	✓
Analytical Tractability	X	✓	✓	✓
Model Accuracy	✓ ✓	✓ ✓	X ✓	✓ X

Main Result

When strategies are comparable in a coordination game, strategy profiles act diffusively near consensus but will form discontinuities determined by the radius or recognition.

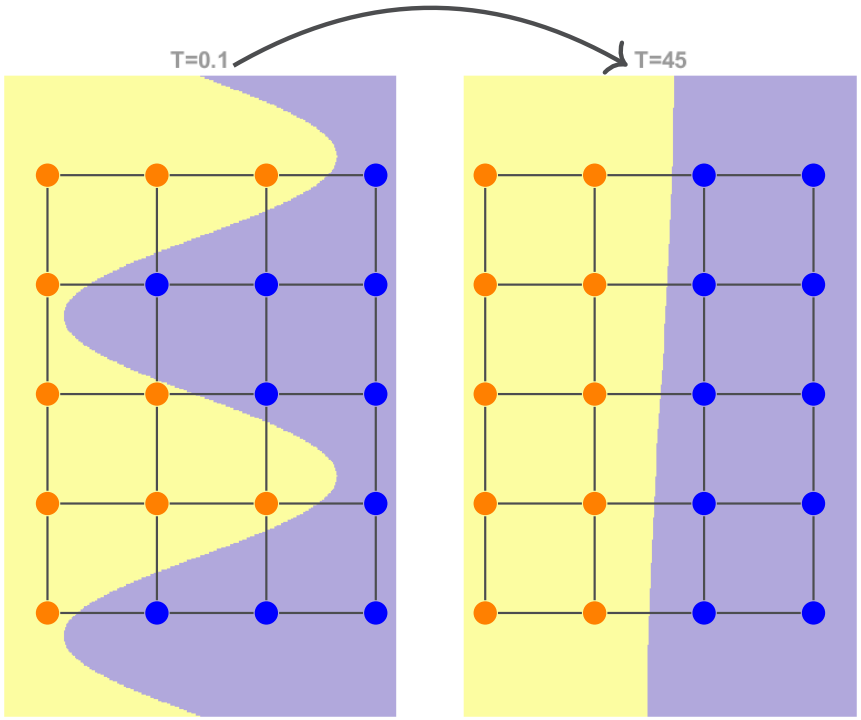
Model Comparison

	Simulation	Dual Model	ODE Model	Nonlocal Model
Main Achievement	Observation	Intuition	Stability	Extension
Numerical Speed	✓	X	✓	✓
Analytical Tractability	X	✓	✓	✓
Model Accuracy	✓ ✓	✓ ✓	X ✓	✓ X

Themes of Coordinating Systems

Diffusion Near Consensus

Models of coordinating systems have a parabolic nature, minimizing an "energy" related strategic gradients



In the dual model, strategic boundaries move in a diffusive parabolic manner

Themes of Coordinating Systems

Diffusion Near Consensus

Models of coordinating systems have a parabolic nature, minimizing an "energy" related strategic gradients

Near Consensus the Replicator model can be written as

$$\frac{d}{dt}u_v^i(t) = u_v^i \cdot \mathcal{L}u^i + (u_v^i)^2 - \langle u_v, g_v \rangle$$

where $\mathcal{L}$ is the graphical Laplacian. Although it is nonlinear, it shares similarities with the graphical diffusion equation.

Themes of Coordinating Systems

Diffusion Near Consensus

Models of coordinating systems have a parabolic nature, minimizing an "energy" related strategic gradients

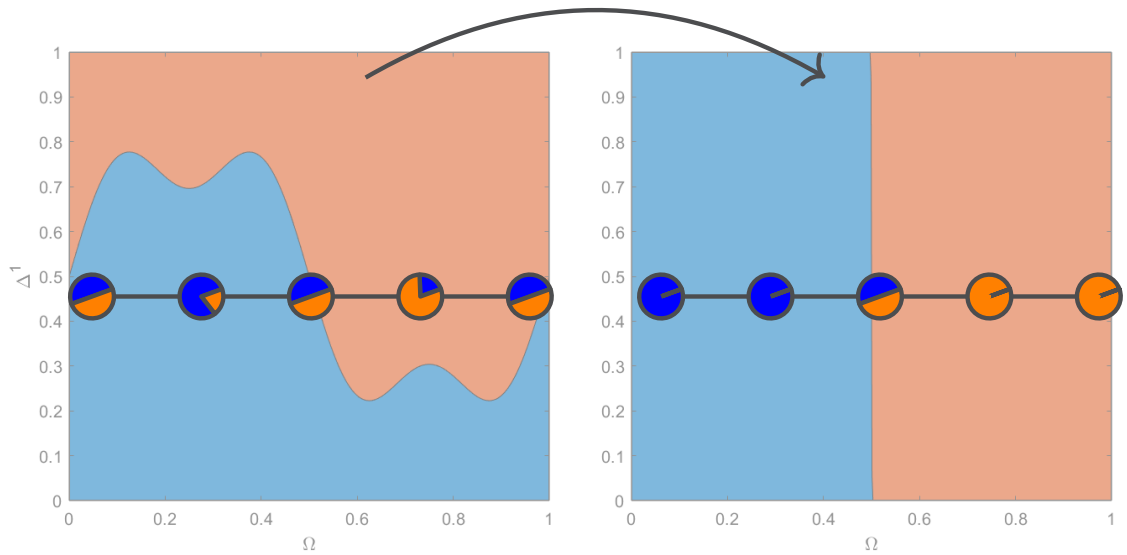
The nonlocal nonlinear diffusion model is obviously diffusive near consensus, when $\omega_u(\Omega) < \delta$ then $\rho'(z) \approx -cz$. This means

$$\int_\Omega K(x,y)\rho'(u(x,t)-u(y,t))dy \approx c \underbrace{\int K(x,y)(u(y,t)-u(x,t))dy}_{=\Delta_K u}$$

Themes of Coordinating Systems

Discretization at a Distance

Models of coordinating systems tend to form discontinuities near regions with large strategic gradients

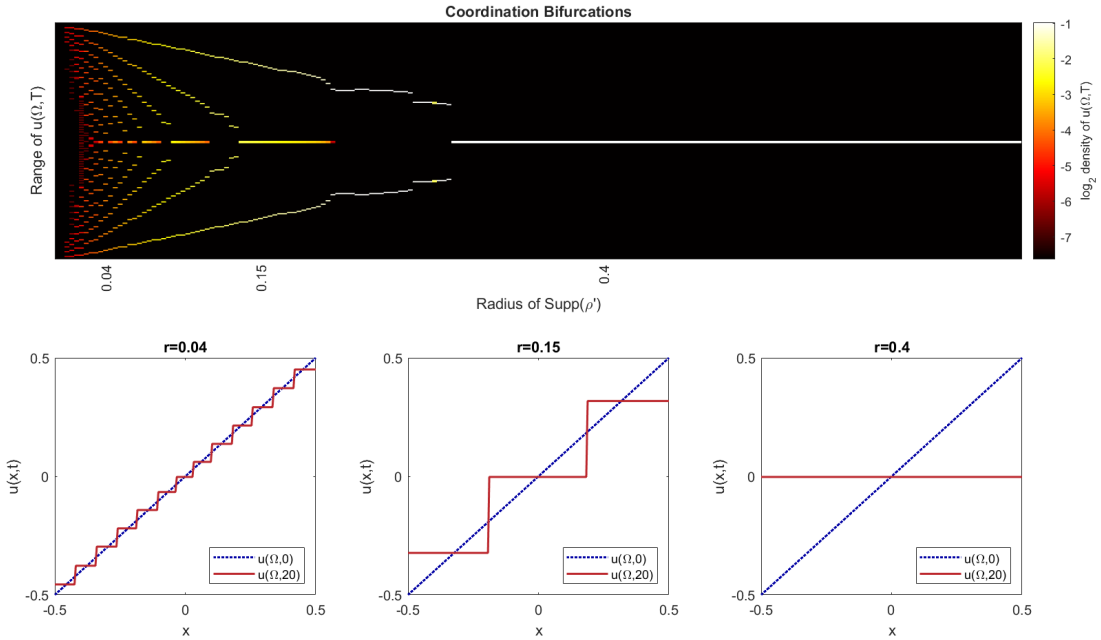


large strategic gradients collapse to discontinuities

Themes of Coordinating Systems

Discretization at a Distance

Models of coordinating systems tend to form discontinuities near regions with large strategic gradients

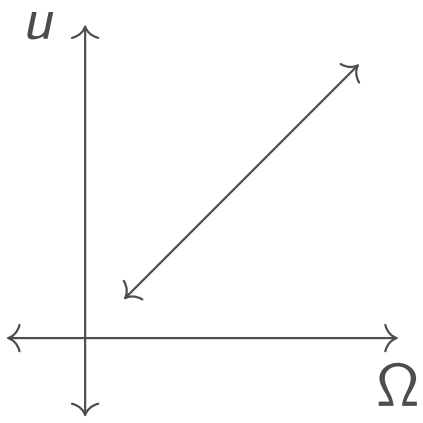


Discontinuities form with predictable regularity when strategies are comparable

Concluding Remarks

Strategic gradient collapse

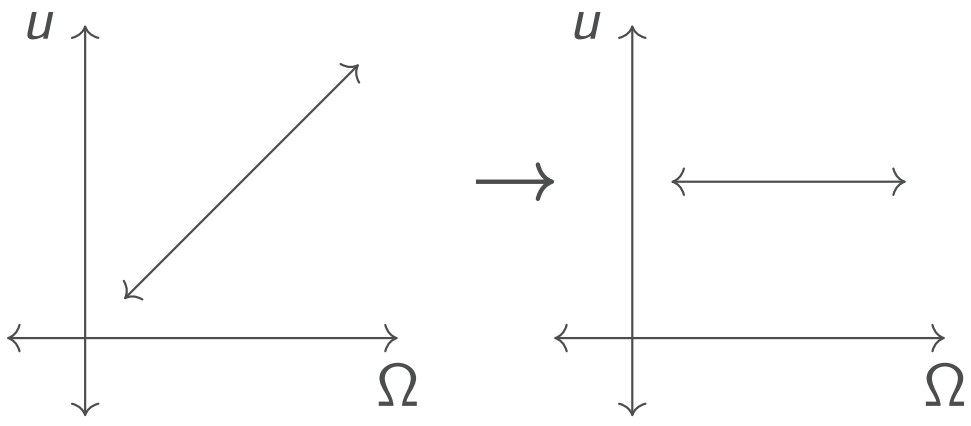
A strategic gradient is, in general, unstable. It can collapse in two mutually exclusive ways.



Concluding Remarks

Strategic gradient collapse

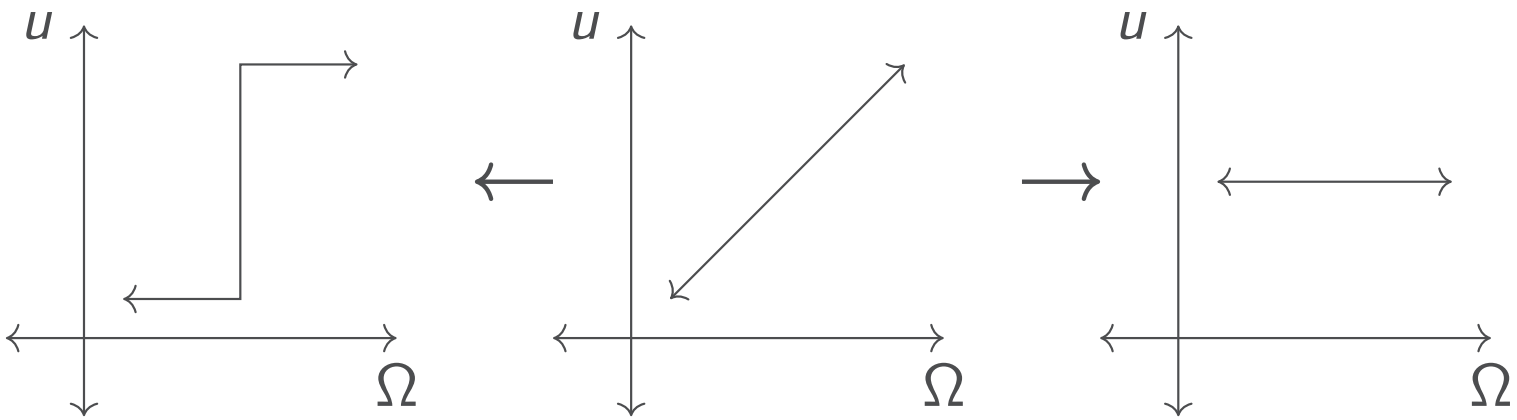
A strategic gradient is, in general, unstable. It can collapse in two mutually exclusive ways.



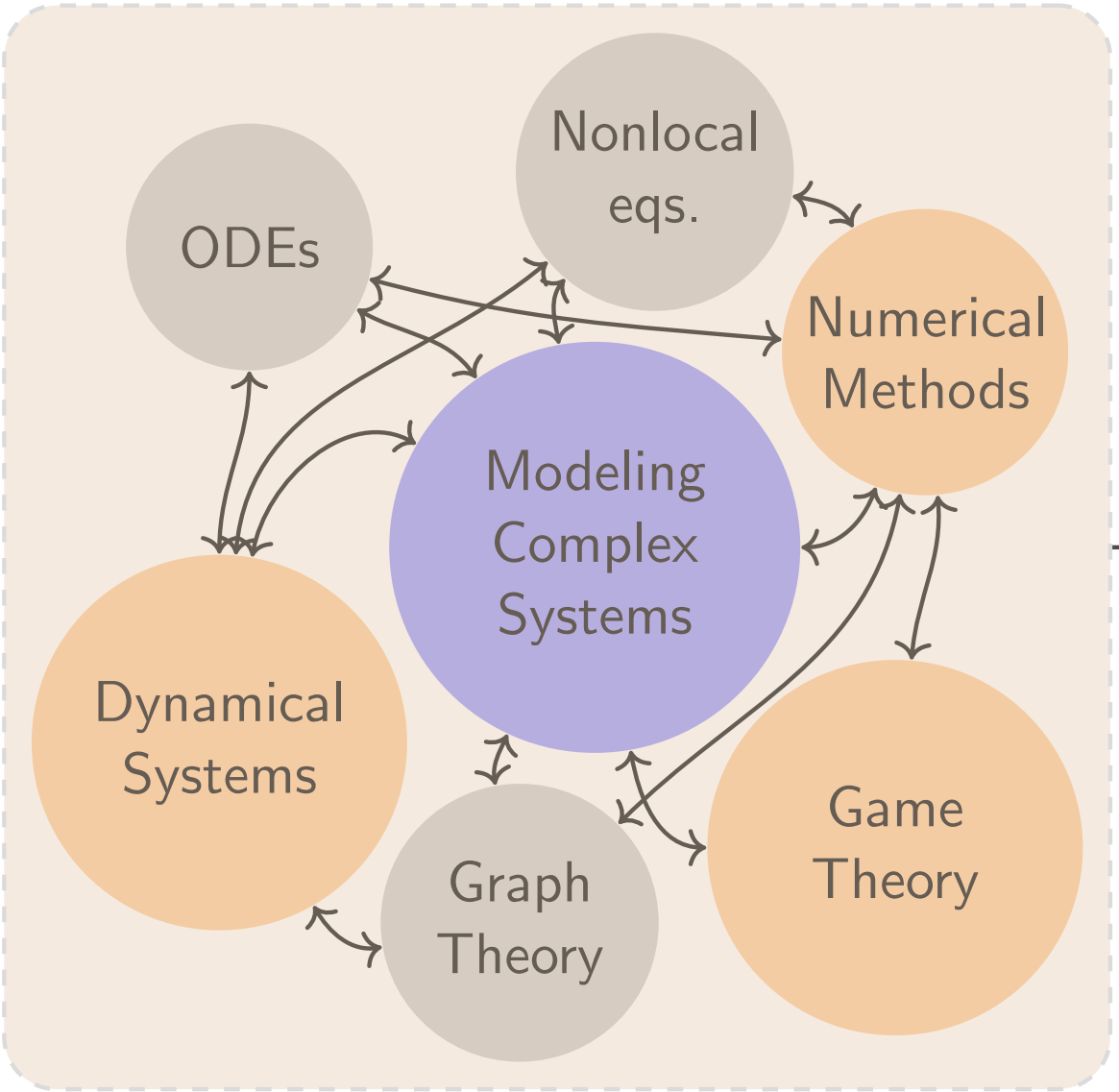
Concluding Remarks

Strategic gradient collapse

A strategic gradient is, in general, unstable. It can collapse in two mutually exclusive ways.



Concluding Remarks

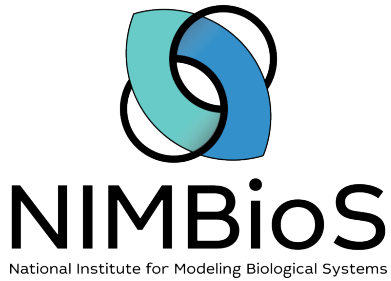


Coordinating systems are diffusive locally and discretizing globally

Acknowledgments

During my time working on these projects, I have been supported by several different grants

- NSF Division of Environmental Biology: Socioeconomic and Epidemiological Drivers of Pathogen Dynamics in Wildlife Trade Networks
- NSF Division of Computing and Communication Foundations: PIPP Phase I: Predicting Emergence in Multidisciplinary Pandemic Tipping Points (PREEMPT)
- NSF Division of Biological Infrastructure: PIPP Phase II: Analysis and Prediction of Pandemic Expansion (APPEX)
- NSF Division of Research on Learning: Developing an Easrly Understanding of Contagion in preschool- and kindergarten-aged Children



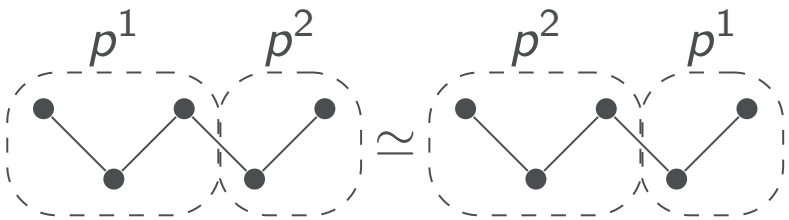
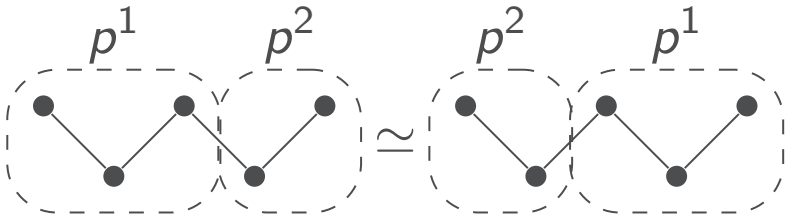
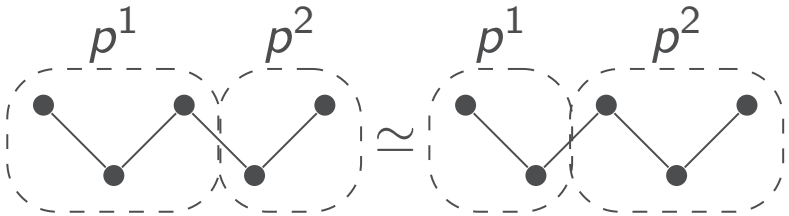
Thank you

Questions?

for publications and code, visit my website

Isomorphism detection

An example of the three different ways two labeled partitions of labeled graphs can be isomorphic to one another and the "vector view," which is simply how the computer stores the partition information.

Type	Partition Diagram	Vector View
Isomorphic by way of relabeling		$[1,1,1,2,2] \simeq [2,2,2,1,1]$
Isomorphic by way of symmetry		$[1,1,1,2,2] \simeq [2,2,1,1,1]$
Isomorphic by way of symmetry and relabeling		$[1,1,1,2,2] \simeq [1,1,2,2,2]$

Isomorphism by way of symmetry

Detecting these isomorphisms just appends an extra step to any graph isomorphism detection algorithm by defining an injective graph expansion.

Graph expansion

For a partitioned labeled graph (G, P) let $\Lambda(G, P)$ be a labeled graph where for every $v \in G$ if $v \in P^k$ then k pendent vertices are added to v.

Λ is an injection from labeled partitions of labeled graphs to labeled graphs. If $\Lambda(G, P) \approx \Lambda(H, Q)$ then $(G, P) \approx (H, Q)$.

Restricting to Two Strategies

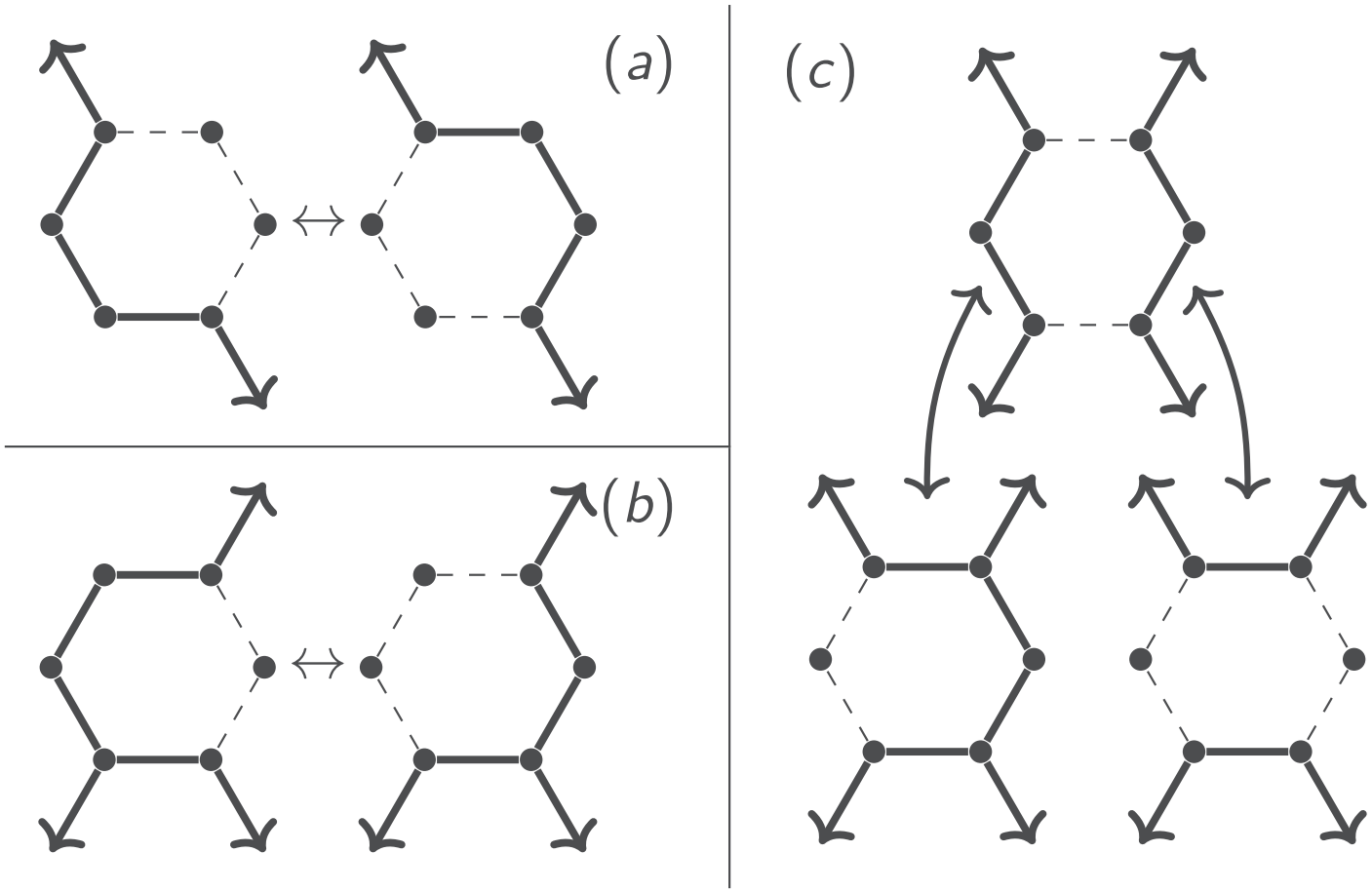
When we restrict only to two strategies both of these issues are solved

- $\Psi : \mathcal{Q}_G^{(2)} \to C(G^*)$ is a bijection where $C(G^*)$ is the cyclespace of G^* and $\mathcal{Q}_G^{(2)}$ is the set of vertex partitions of G with two parts.
- The cycle space is a vector space over $\mathbb{Z}_2$

Helpful features of the two strategy case

This result comes from the fact that a partition into two parts is determined exactly by its cutset and that the cutspace of G and the cyclespace of G^* are isomorphic.

Examples of SFRs



Restricting to Two Strategies

When we restrict only to two strategies both of these issues are solved

- $\Psi : \mathcal{Q}_G^{(2)} \to C(G^*)$ is a bijection where $C(G^*)$ is the cyclespace of G^* and $\mathcal{Q}_G^{(2)}$ is the set of vertex partitions of G with two parts.
- The cycle space is a vector space over $\mathbb{Z}_2$

Theorem 3.4

u is a Nash equilibrium with two strategies if and only if $s = \Psi \circ \Phi u$ satisfies

$$size(s) \leq size(s + C_f) \quad \text{for any face } f$$

Equilibrium Results in the Two Strategy Case

Cyclespace

For a graph G the cyclespace is the set of all subgraphs of G^* in which every vertex has even degree (it admits an Eulerian circuit). This is a vector space over the field $\mathbb{Z}_2$.

s and r are adjacent in $(C(G^*), D)$ if $s = r + C_f$ where C_f is the cycle of edges adjacent to the face f.

Theorem 3.4

u is a Nash equilibrium with two strategies if and only if $s = \Psi \circ \Phi u$ satisfies

$$size(s) \leq size(s + C_f) \quad \text{for any face } f$$

Cycle Shortening Flow

We can also construct a flow which is defined entirely on $C(G^*)$ which is equivalent to the MBR

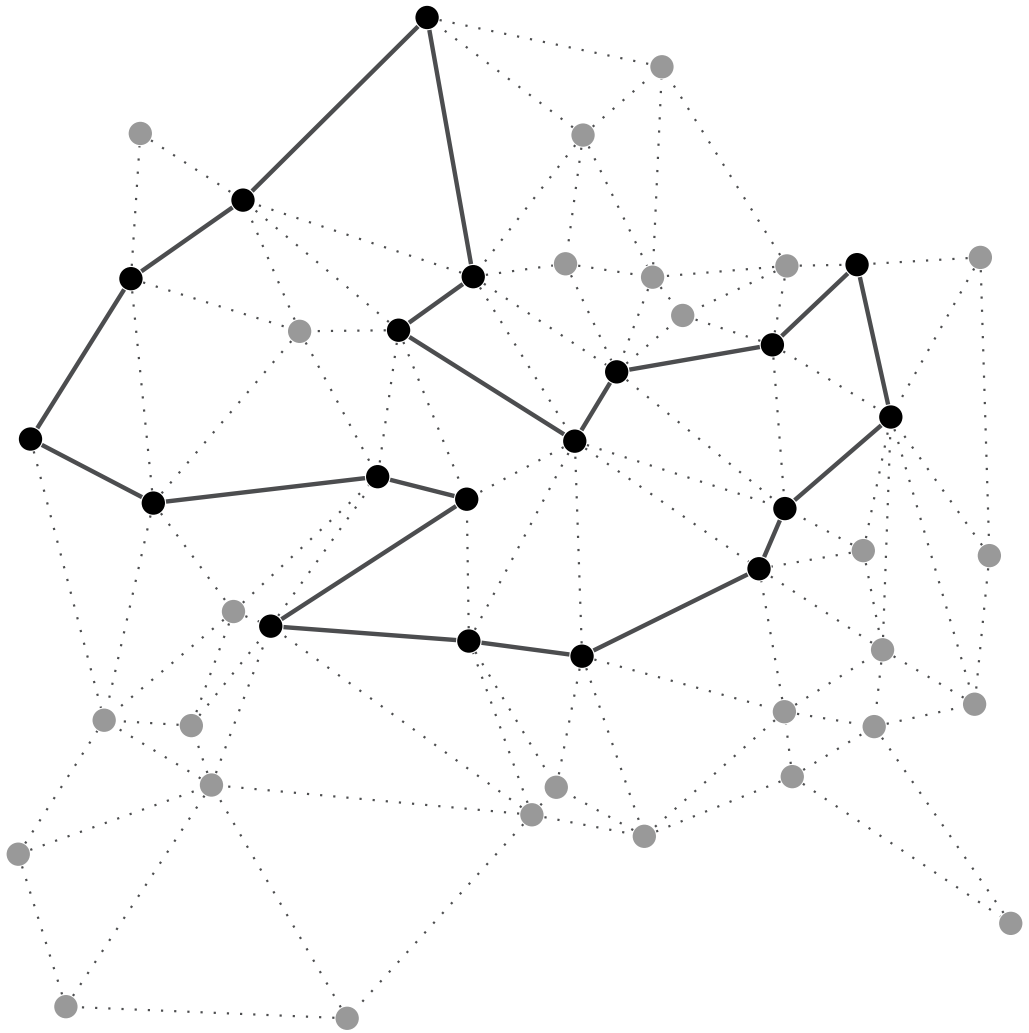
Cycle Shortening Flow

A sequence of subgraphs satisfies the cycle shortening flow if

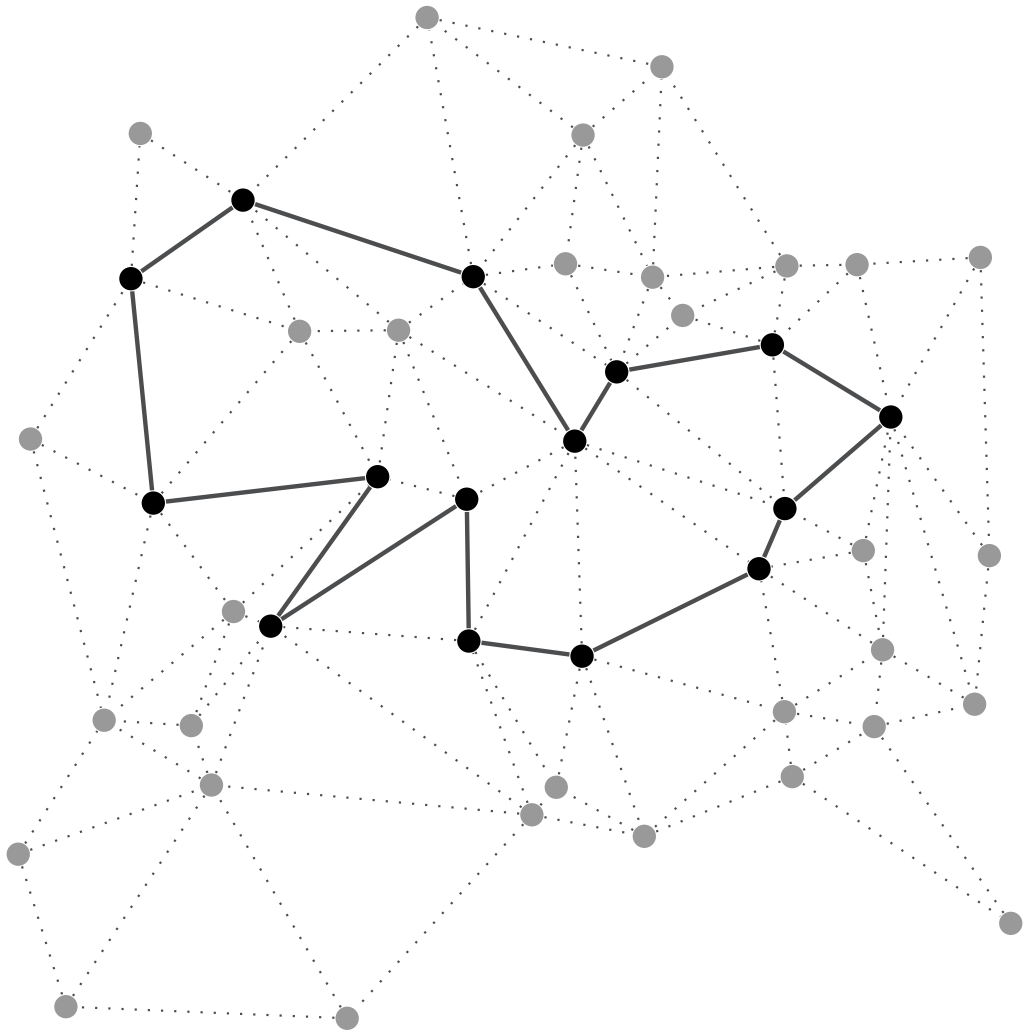
$$s(t+1) = s(t) + \sum_{f \in F} H_f(s(t)) C_f$$

where $H_f(s(t)) = 1$ if at more than half of the edges incident to f are included in $s(t)$ and 0 otherwise.

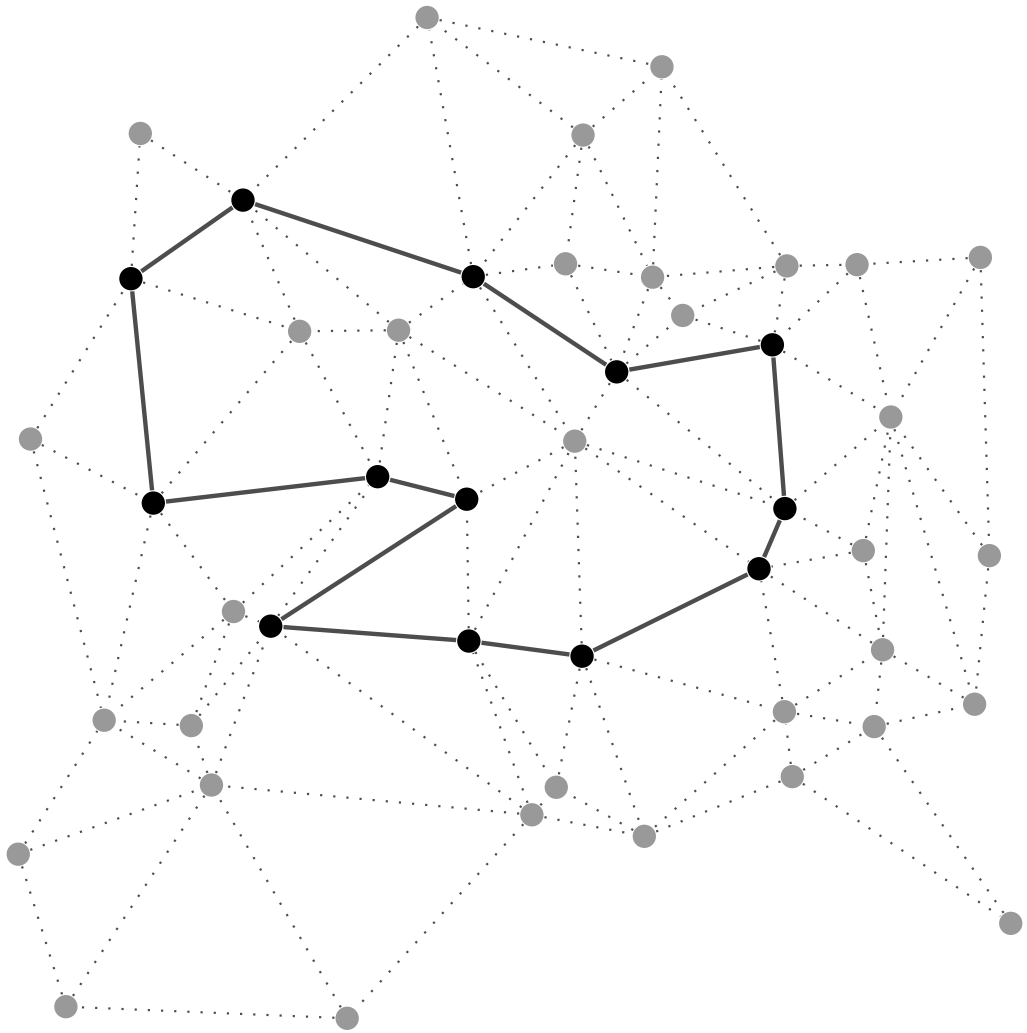
Example



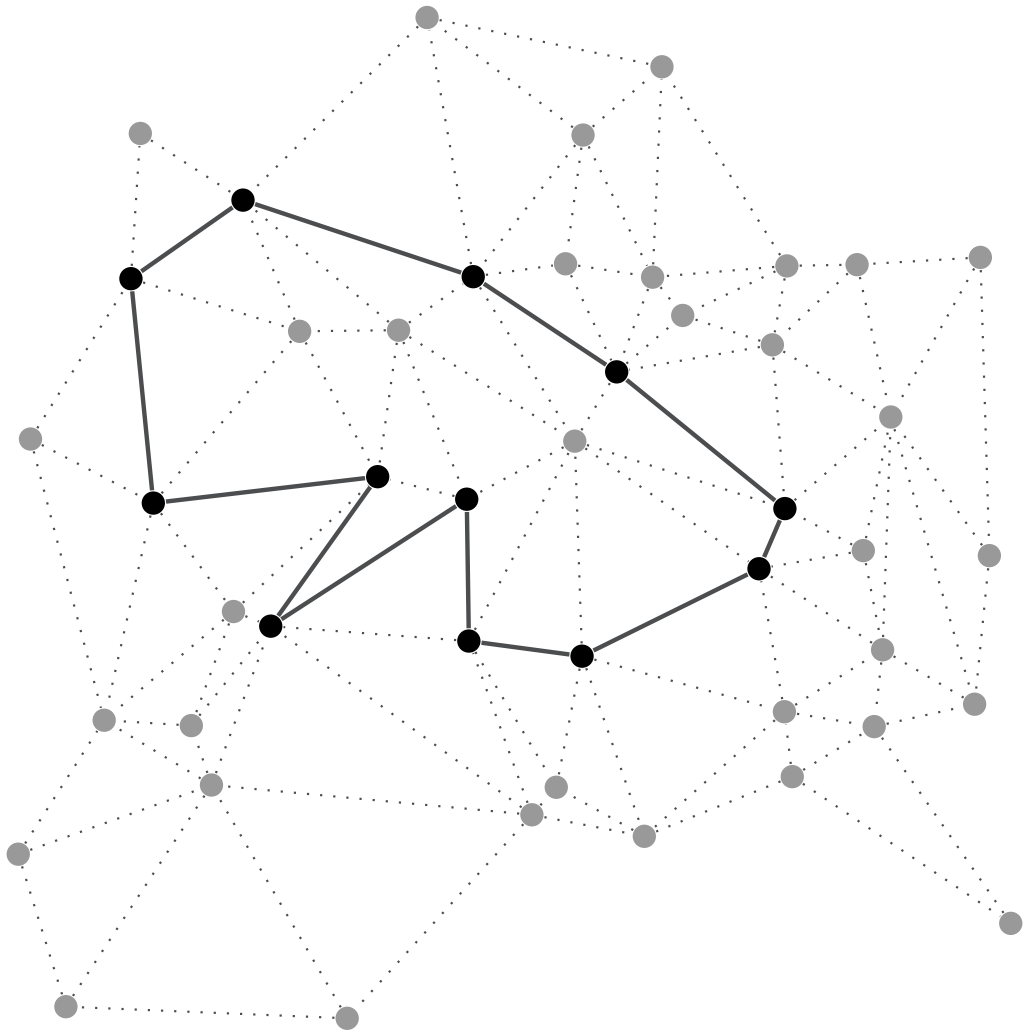
Example



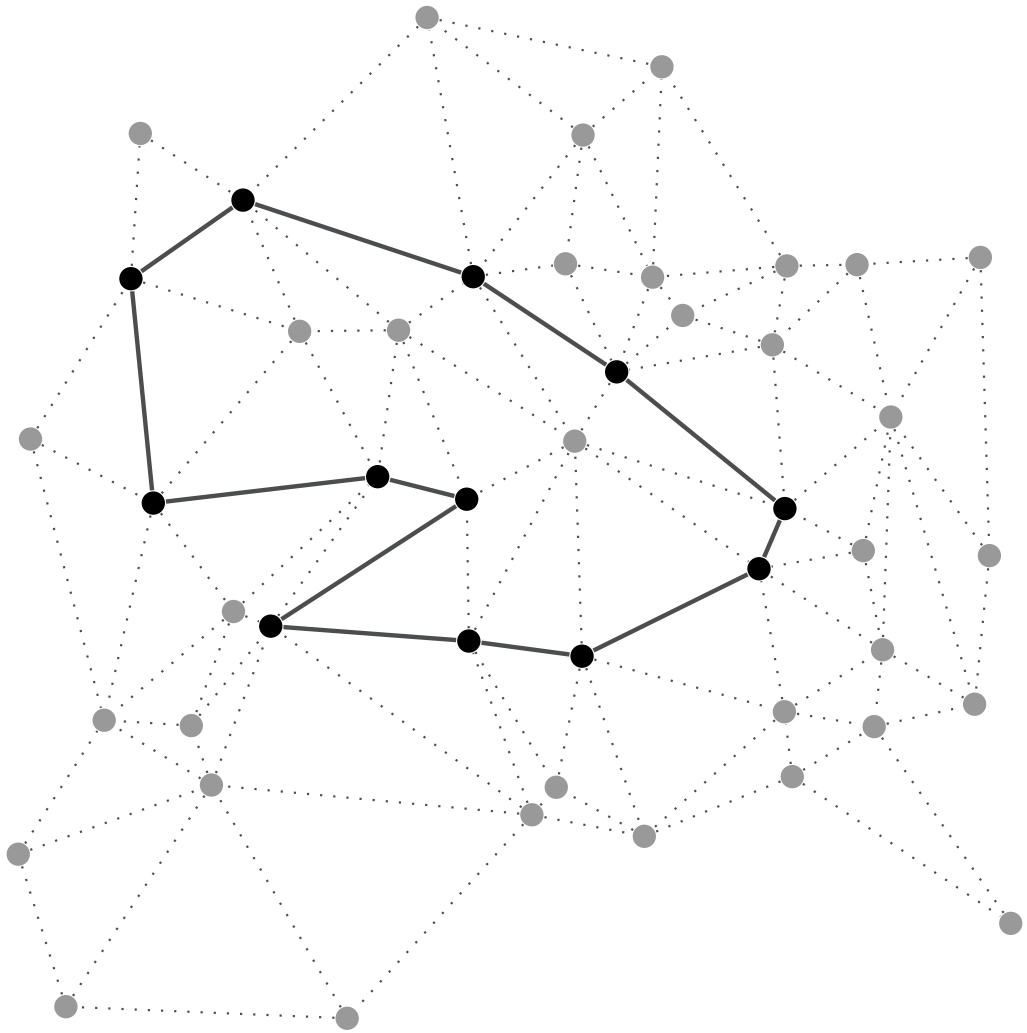
Example



Example



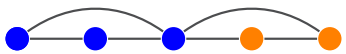
Example



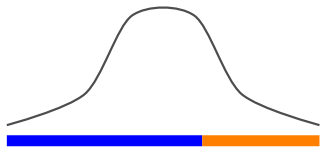
Continuous Extension

Not only is this helpful for the analysis of coordinating systems in planar graphs, it leads us to exciting extensions with continuous player spaces

- Suppose that every player is a point in $\Omega \subseteq \mathbb{R}^2$
- Suppose each player interacts with other players according to the kernel $K \in C_b^0(\Omega; L^1(\Omega))$.
- Fitness for playings strategy i is computed as $g^i[u](x) = \int_\Omega K(x,y)u^i(y)dy$



v



x

$w_v(\bullet, u) = \sum_{w \in V} W_{w,v}\langle \bullet, u_w \rangle$ $\qquad$ $w_v(\bullet, u) = \int_\Omega K(x,y)\langle \bullet, u(y) \rangle dy$

MBR recapitulates Mean Curvature Flow

Theorem 3.6

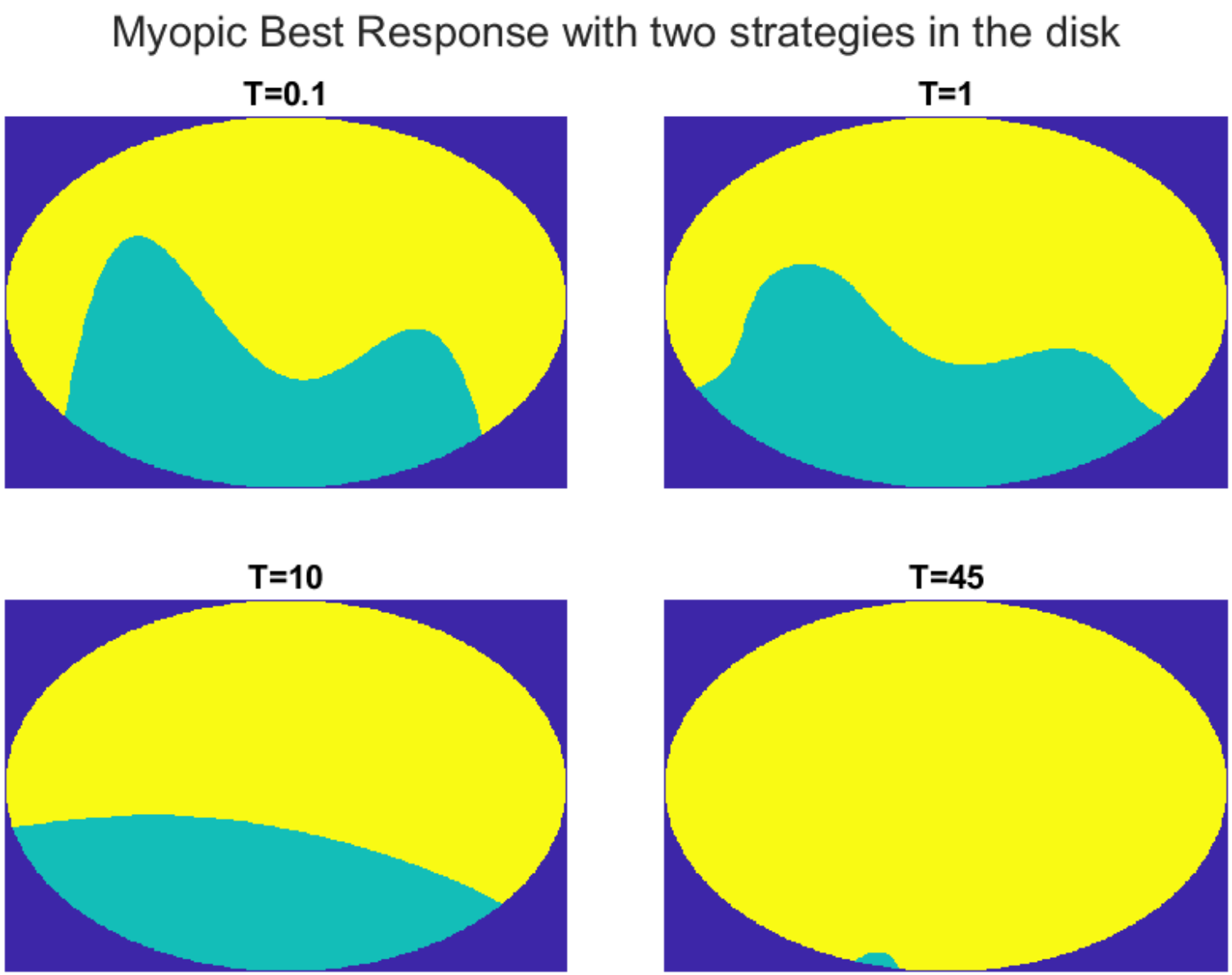
Consider an initial strategy profile $u_0 : \mathbb{R}^n \to C$ is given as $U_0^2 = (U_0^1)^c$ so that $\partial U_0^1 = \partial U_0^2$. Let payoffs be given by equation

$$w_x(i|u) = \int_\Omega K(x,y)u^i(y)dy$$

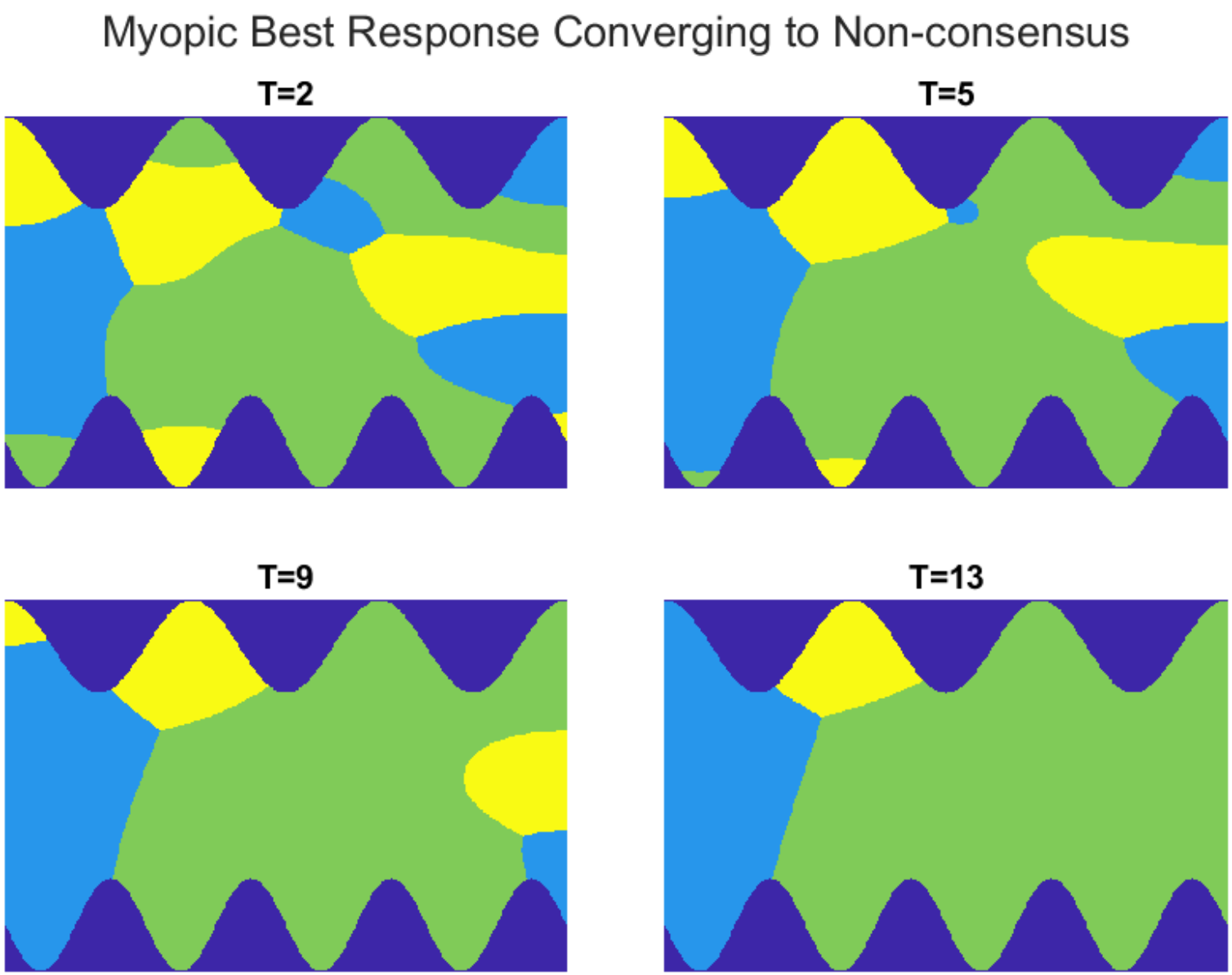
with the kernel $K_\epsilon(x,y) = (4\pi\epsilon)^{\frac{-n}{2}} e^{\frac{-|x-y|^2}{4\epsilon}}$. Under the myopic best response process with time step $\Delta t = \epsilon^2$, the strategic boundary ∂U^1 will move according to mean curvature flow in the limit as $\epsilon \to 0$

The proof is surprisingly direct. The myopic best response time stepping process is equivalent to a time stepping approach from Bence et al. (1992) which Evans and Spruck proved to converge to MCF in 1993. This is equivalent to a argument by Caffarelli from a threshold dynamics setting.

Continuous Example



Continuous Example



Structured Replicator Dynamics

Imagine each vertex in the graph G as an infinite patch, we abstract this to say that each vertex is itself one player with a mixed strategy

Strategy Space

With the inclusion of mixed strategies our strategy space is

$$(\Delta^{m-1})^V$$

Each player has a strategy in Δ^{m-1} so a strategy profile can be thought of as a collection of m-dimensional vectors.

Payoff Function

For any normal form game with payoff matrix A, the payoffs for player v are computed by using the following shorthand

$$g_v = \sum_{w \in V} W_{v,w} A u_w$$

Benefits of the ODE system

Lemma 4.2

$(\Delta^{m-1})^V$ is an invariant manifold for the ODE system

Corollary 4.1

Solutions to the IVP with initial data in $(\Delta^{m-1})^V$ will exist and be unique for all time.

Lemma 4.3

If $u(t)$ solves the IVP with $u(0) \in (\Delta^{m-1})^V$ then

$$\frac{d}{dt}\bigg|_{\zeta=0} w_v(u_v + \zeta \frac{d}{dt} u_v | u) \geq 0$$

with equality only when $\frac{d}{dt} u_v = \vec{0}$. This means the replicator dynamics represent a *Better Response* strategy revision protocol.

Better Response

Being a better response strategy revision protocol still has many benefits.

Lemma 4.4

For a potential game with potential function $\mathcal{W}$, if $u(t)$ solves the ODE system then $\frac{d}{dt}\mathcal{W}(u(t)) \geq 0$ for all time with equality only when $\frac{d}{dt}u(t) = \vec{0}$.

Theorem 4.1

If $u(t)$ is a convergent trajectory in the interior of $(\Delta^{m-1})^V$ then $\lim_{t\to\infty} u(t)$ exists and is a Nash equilibrium.

However, the fact that it is a better response strategy means that the dynamic behavior of the system will not always coincide with what we expect from the discrete time system.

What can the system tell us about the game

To ensure that it is a reasonable model in the evolutionary game theoretic sense, we must ensure that it satisfies the three conditions of the Folk Theorem of Evolutionary Game Theory

Folk Theorem of Evolutionary Game Theory (Cressman and Apaloo)

a. A stable rest point is a Nash equilibrium
b. A convergent trajectory on the interior of the domain evolves to a Nash equilibrium
c. A strict Nash equilibrium is locally asymptotically stable.

What can the game tell us about the behavior of the system

For the pure coordination game in particular, we can get close to a characterization of dynamics near equilibria

Theorem 4.4

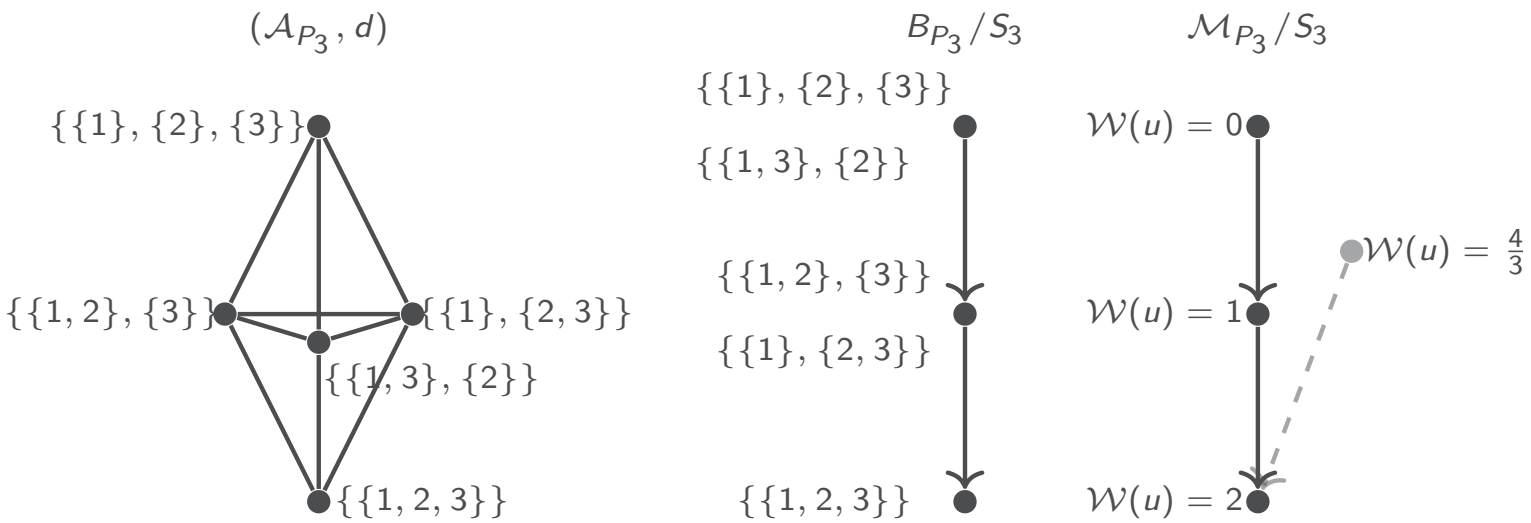
Let $u^\star$ be a strategy profile to the pure coordination game played on the graph G the behavior of the ODE system in the vicinity of equilibria can be characterized in the following way

- If $u^\star$ is a strict Nash equilibrium, then it is locally asymptotically stable.
- If $u^\star$ is a Nash equilibrium but not a strict Nash equilibrium and it satisfies condition (A) then it is stable in the Lyapunov sense
- If $u^\star$ is a Nash equilibrium but not a strict Nash equilibrium and it satisfies condition (B) then it is unstable in the Lyapunov sense
- If $u^\star$ is not a Nash equilibrium then it is unstable

Morse Graph Relationship

Morse Graphs for Potential Games

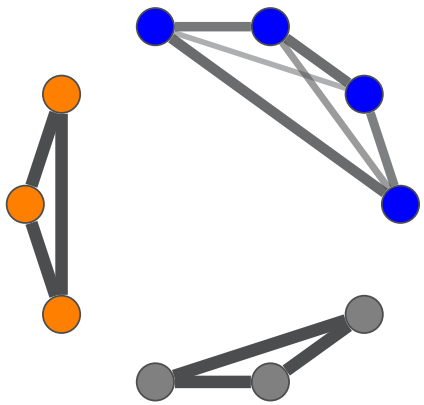
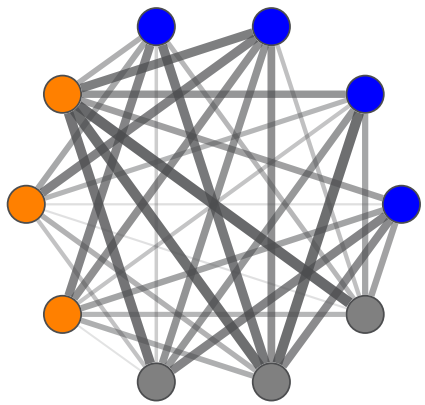
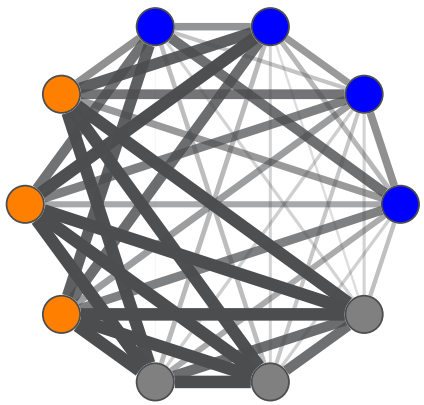
The Morse graph of the ODE system is very closely related to the strategy space in the discrete time game.



Connection to the discrete time game

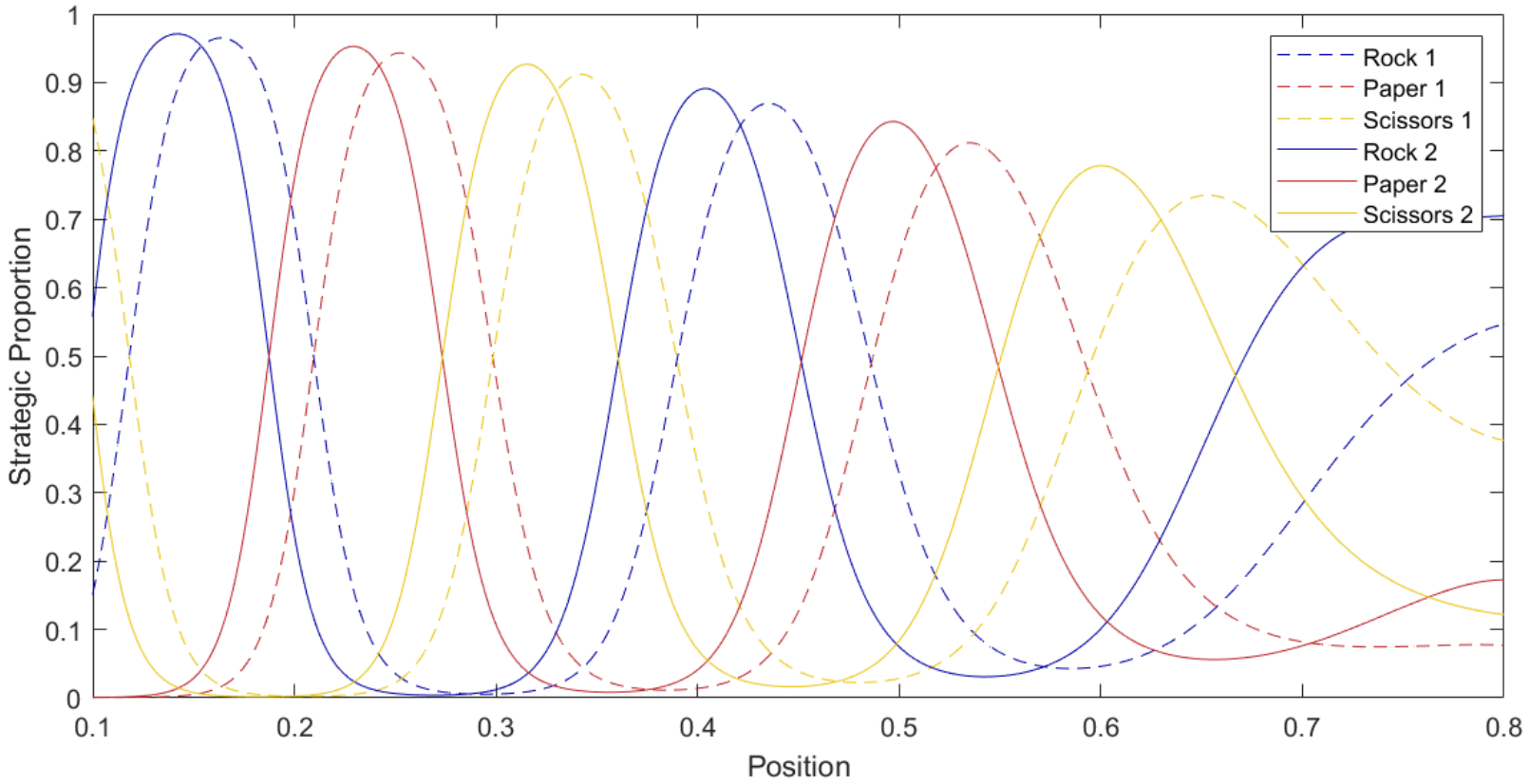
This gives us a stability criterion which we can apply in several ways

Stability Optimization

A	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 10 & 1 & -10 \\ 1 & 0 & 1 \\ -10 & 1 & 10 \end{bmatrix}$
$W^\star$			
λ_0	-2	0	-5.46

Continuous Extensions

Another exciting element of this model is that it is easily extendable to a continuous player.

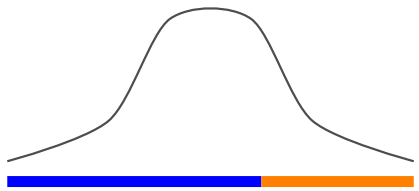


Continuous Player Space

Continuous Player Space

- Every player $x \in \Omega \subseteq \mathbb{R}^n$
- Two players interact with each other according to the kernel $K \in C_b^0(\Omega; L^1(\Omega))$ (continuous and bounded w.r.t. to the first variable in the L^1 sense, integrable w.r.t. the second).
- A player's payoff is determined by integrating against the kernel

$$w(x|u) = \int_\Omega K(x,y)\rho(u(x) - u(y))dy$$



x

$$w_v(\bullet, u) = \int_\Omega K(x,y)\rho(\bullet - u(y))dy$$

Themes Regarding Coordination

Coordination, in general, is the result of two competing forces. Diffusion near equilibrium and discretization at a distance

Diffusion near equilibrium

Each of our models demonstrates diffusive action. For example:

- In the Dual model, strategic boundaries move in a parabolic manner as in threshold dynamics
- In the nonlocal model, close to equilibrium $g[u] \approx \Delta^s(u)$

Discretization at a distance

- In the replicator model, numerical simulations show a discretization as solutions tend towards equilibria
- In the nonlocal model, stationary solutions provably have discrete images.