On Structured Models of Coordinating Systems

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Dissertation Defense
University of Tennessee - Knoxville

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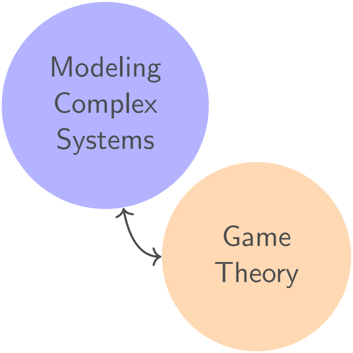
THE UNIVERSITY OF TENNESSEE KNOXVILLE
DEPARTMENT OF MATHEMATICS

Overview

Modeling
Complex
Systems

Overview



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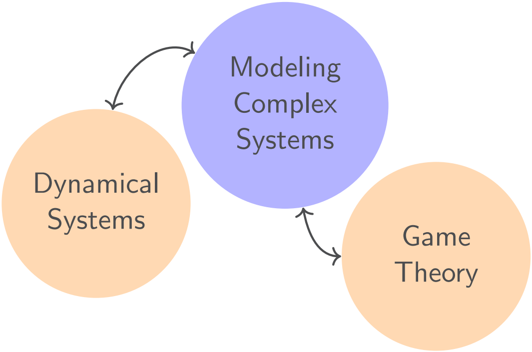

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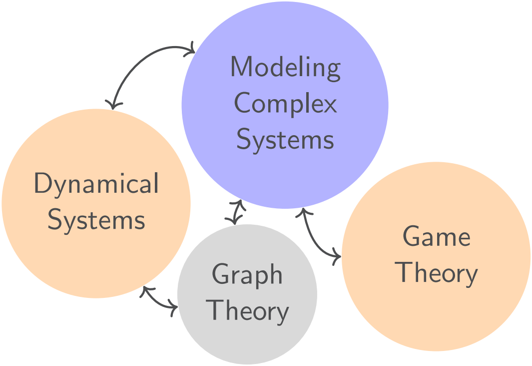
Modeling Complex Systems
Dynamical Systems
Graph Theory
Game Theory

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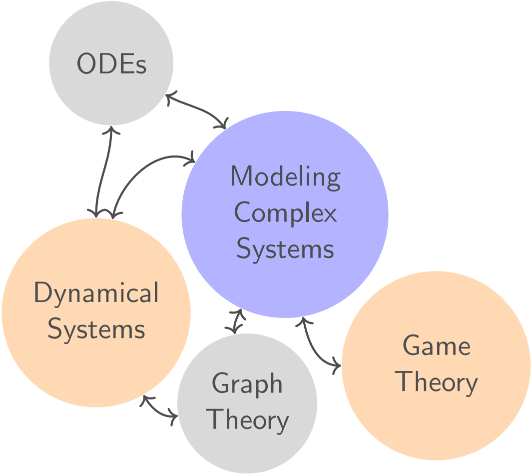
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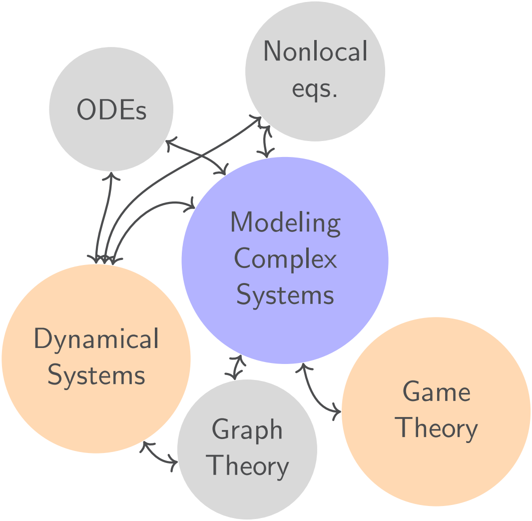

ODEs
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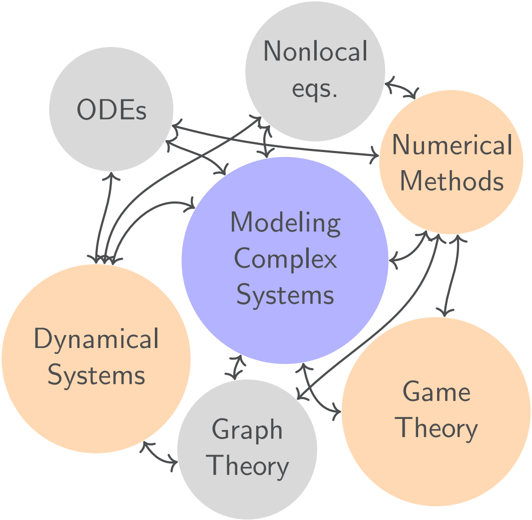

Nonlocal
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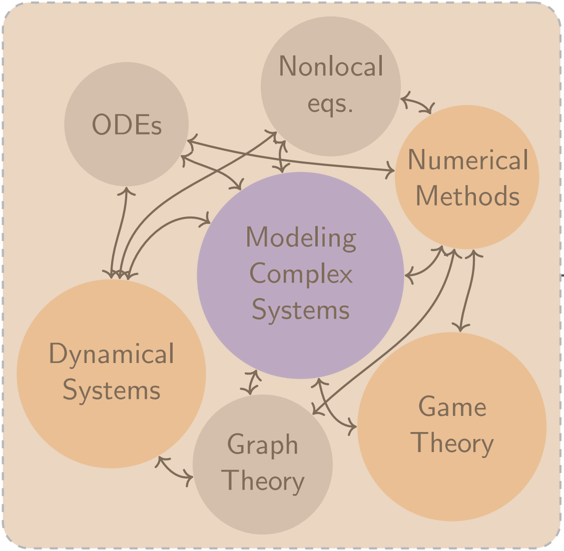

Can we predict how coordinating systems will organize themselves?

Overview

Introduction

Coordination

A fundamental game theoretic interaction in which two players receive a higher payoff by playing the same strategy than they would by playing different strategies.

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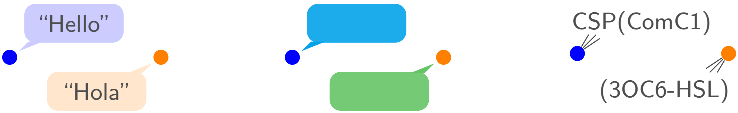

Definitions

To give a more rigorous description of the game we need the following definitions. For a game with players V and strategy set C:

- $u : V \to C$ is a strategy profile. describes which players play which strategies
- $w_v(c|u)$ is the Payoff for player v playing strategy c against the strategy profile u
- $BR_v(u) = \text{argmax}_{c \in C}\, w_v(c|u)$ is player v's best response to u
- u is a Nash equilibrium if $u_v \in BR_v(u)$ for all $v \in V$.

Bandwagon property

Bandwagon Property (Kandori et al. 1993)

A game is a coordination game if and only if it satisfies the bandwagon property: for any strategy profile u,

$$BR_v(u) \subseteq C(u) \quad \forall v \in V$$

where $C(u)$ is the support of a strategy profile u (i.e. the set of all strategies being used)

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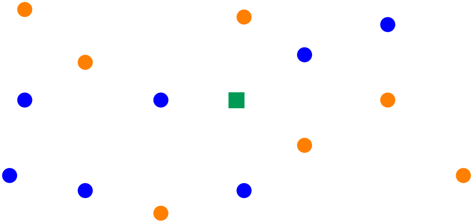

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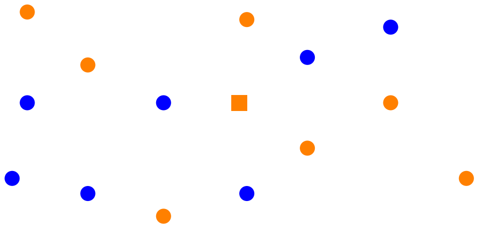
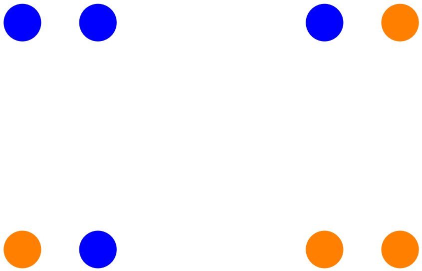
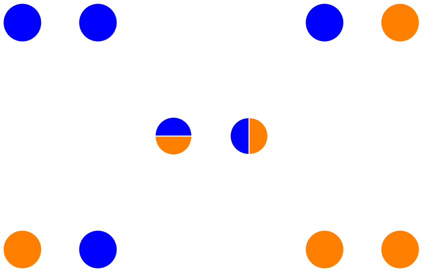


Two player coordination game



The two player coordination game is entirely solved. There are three Nash equilibria. Two pure strategy Nash equilibria (consensus) and one mixed strategy Nash equilibrium (mixed consensus)

Two player coordination game



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Two player coordination

The diagram encodes the payoff matrix for any coordination game

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where $a > c$ and $d > b$.

Using this to describe the two player coordination game more rigorously:

- $V = \{1, 2\}$
- $C = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$
- $w_v(u_v|u_w) = u_v^\intercal A u_w$ (and so if $A = I_2$ then $w_v(u_v|u_w) = \langle u_v, u_w \rangle$)

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n-player coordination

The n-player payoff function is the sum of the pairwise payoffs

$$w_v(u_v|u) = \sum_{w \in V} u_v^\intercal A u_w$$

Dynamic coordination

To understand the systems dynamically, we will use the Myopic Best Response (MBR) strategy revision protocol.

Myopic Best Response

A sequence of strategy profiles $u(t)$ satisfies the myopic best response strategy revision protocol if, for all t,

$$u_v(t+1) \in BR_v(u(t)) \quad \forall v \in V$$

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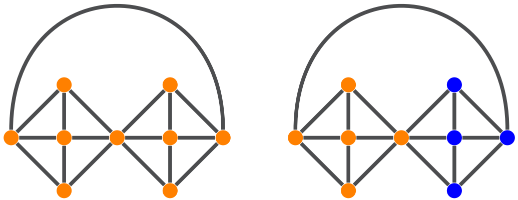
MBR sequences

If a MBR sequence converges to a limit, its limit is a Nash equilibrium.

The structured coordination game

For the Structured Coordination Game played on the graph G with adjacency matrix W, the payoff function is

$$w_v(u_v|u) = \sum_{w \in V} W_{v,w} \langle u_v, u_w \rangle$$



Consensus is always a Nash equilibrium, but some graphs admit non-consensus Nash equilibria.

The structured coordination game by other names

Central Question

Given some graph, what non-consensus equilibria does it admit?

Many domains use a different name to refer to the same type of question. This itself is a coordination game.

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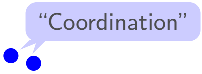

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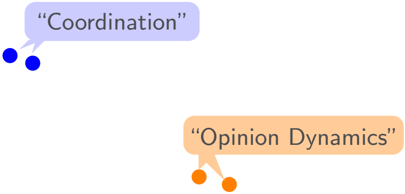

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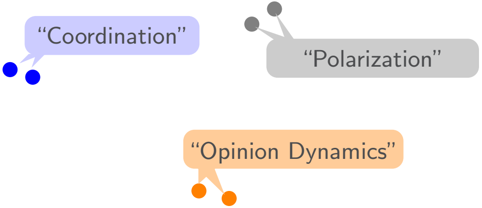

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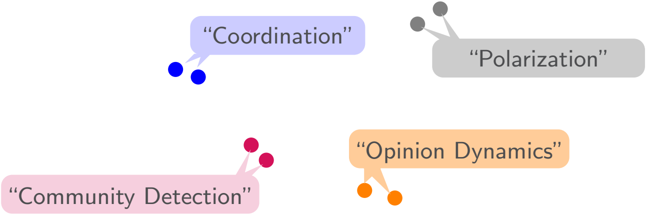

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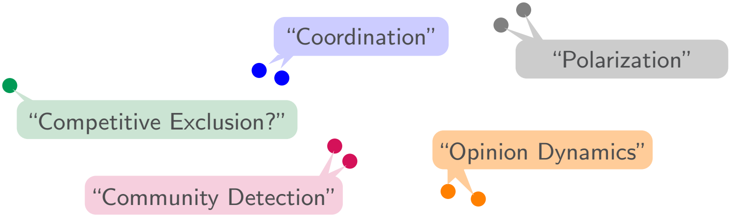

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Modeling challenges

Every attempt to model a coordinating system must contend with these three challenges:

Numerical Speed	Can the model work quickly in large domains?
Analytical Tractability	Can we use the model to prove general results?
Model Accuracy	Does the model use best response dynamics? Does the model use the correct payoff function?

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Model Diversity

No model will be able to overcome all of these challenges, but through a wide range of models we can improve our understanding of coordinating systems.

Simulation study

Our first modeling approach is through simulation

Dynamic Games and Applications
https://doi.org/10.1007/s13235-024-00612-4

Insights into the Structured Coordination Game with Neutral Options Through Simulation

John S. McAlister[1,3] · Nina H. Fefferman[1,2,3]

Accepted: 30 November 2024

Simulation set up

We focus on the setting where $A = I_m$ where m is the number of strategies available.

Payoff Function

$$w_v(u_v|u) = \sum_{w \in V} W_{v,w} \langle u_v, u_w \rangle$$

Strategy Revision Protocol

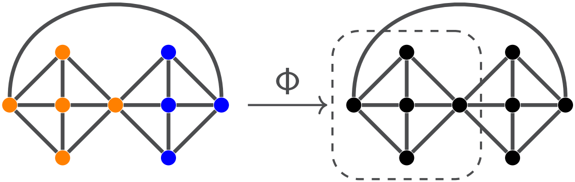
$$u_v(t+1) \in BR_v(u(t))$$

If $|BR_v(u(t))| > 1$ and $u(t) \in BR_v(u(t))$ then $u(t+1) = u(t)$ it $u(t) \notin BR_v(u(t))$ then $u(t+1)$ is drawn uniformly randomly.

Partitioning approach

Strategy Space

Let X_G be the set of all strategy profiles on G and let $\mathcal{Q}_G$ be the set of all vertex partitions of G. Define $\Phi : X_G \to \mathcal{Q}_G$



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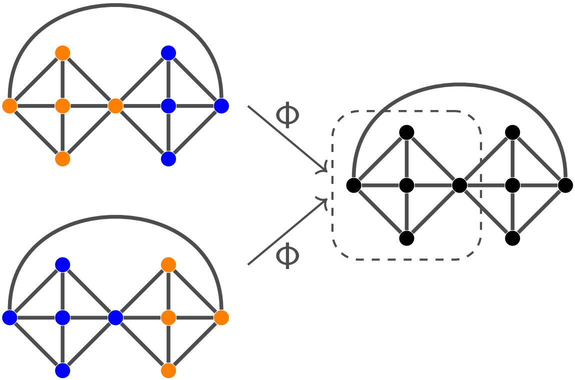

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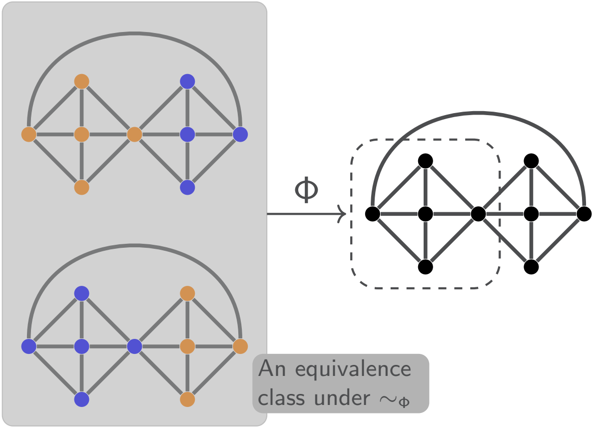

Reducing the state space

Reduced Strategy Space

Define $\mathcal{A}_G$ which is the set of equivalence classes of X_G under Φ.

$$\mathcal{A}_G = X_G / S_m$$

Equilibrium partition

An equilibrium partition is a partition P such that $\Phi^{-1}P \in \mathcal{A}$ is a Nash equilibrium.

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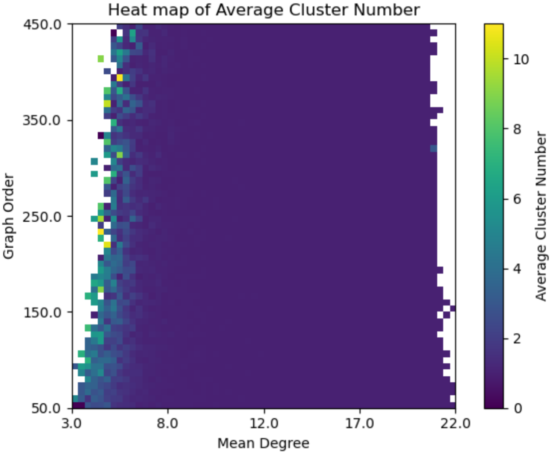
With this approach we simulate several million Myopic Best Response sequences through the strategy space for several thousand random graphs and consider their limits.

Observations from the simulation

Non-consensus equilibria only appear in the transitional region of connectedness in the E-R sense.



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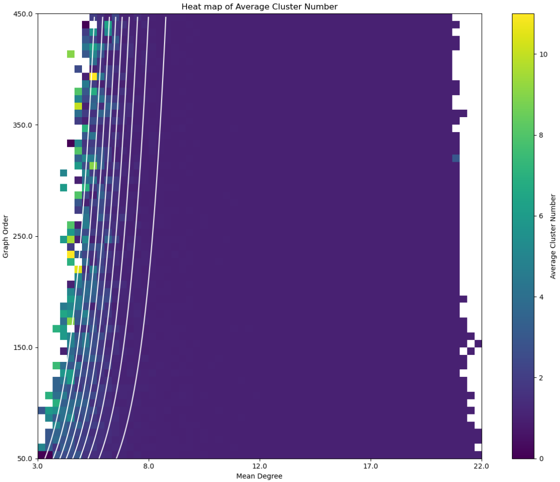

Results of the simulation

	Simulation
Main Achievement	
Numerical Speed	
Analytical Tractability	
Model Accuracy	

Results of the simulation

	Simulation
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Results of the simulation

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Main Achievement	Observation
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Results of the simulation

	Simulation
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Numerical Speed	✓
Analytical Tractability	X
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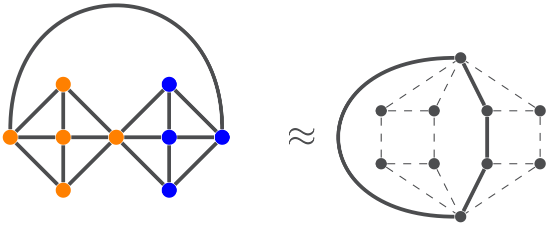
Main result

Non-consensus equilibria occur in the transitional area of connectedness, when graphs are "minimally connected."

Minimal subgraphs

To try to achieve more analytical power we will attempt to consider the structured coordination game in the dual sense.



Potential games

Potential Game (Monderer and Shapley 1996)

A potential game is any game which admits a function $\mathcal{W}$ so that for two strategy profiles $u = (u_v, u_{-v})$ and $u' = (u'_v, u_{-v})$ then

$$\mathcal{W}(u) - \mathcal{W}(u') = w_v(u_v|u) - w_v(u'_v|u')$$

Lemma 3.2

$\mathcal{W}(u) = \frac{-1}{2}E(\Phi u)$ is a potential function for the pure coordination game where $E(P)$ is the size of the cutset for the partition P.

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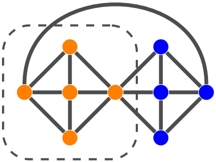
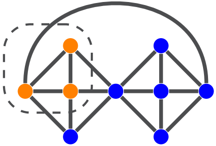
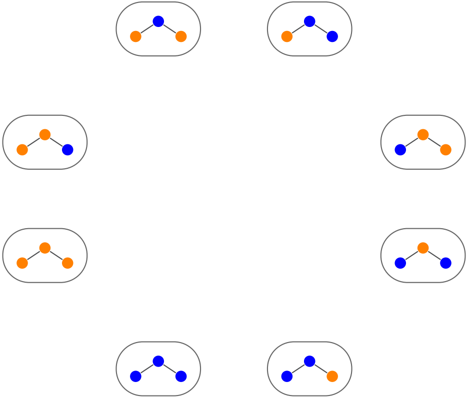
$\mathcal{W}(u) = -2$





$\mathcal{W}(u) = \frac{-5}{2}$

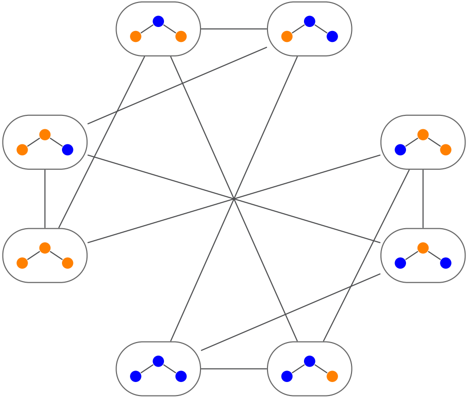
Defining locality



X_{P_3}

The set of all strategy profiles on the graph P_3 (here only the strategy profiles with 2 strategies are pictured).

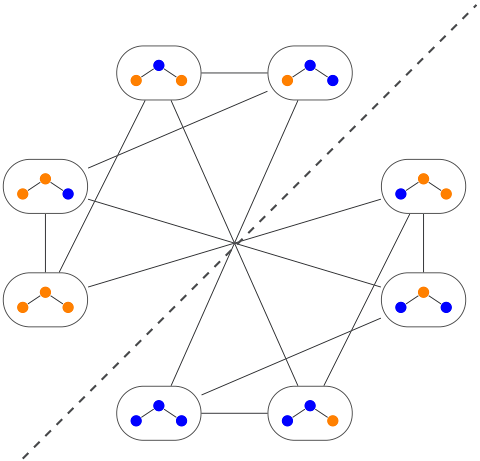
Defining locality



(X_{P_3}, d)

Two strategy profiles are adjacent if they are separated by a single strategic change. This adjacency defines a metric d.

Defining locality



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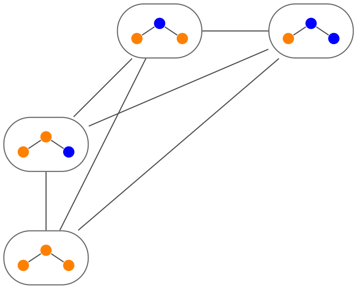
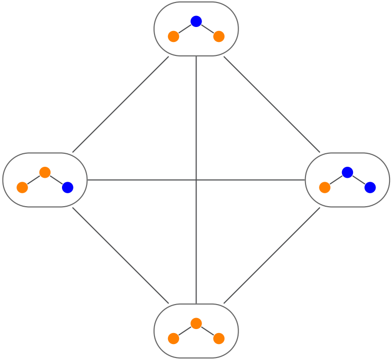


$(\mathcal{A}_{P_3}, d)$

In the same way as before, we reduce the metric space to equivalence classes under $\sim_\Phi$. Each edge is a single strategic change and (maybe) a strategic permutation

Defining locality



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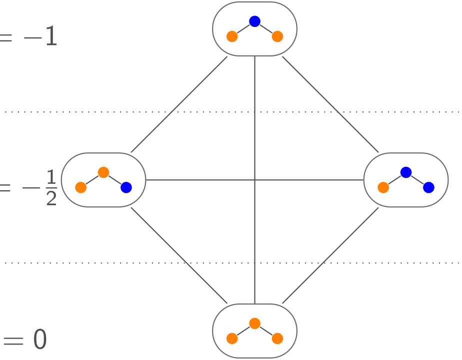
Defining locality

$(\mathcal{A}_{P_3}, d)$

$\mathcal{W} = -1$

$\mathcal{W} = -\frac{1}{2}$

$\mathcal{W} = 0$



Each element of $\mathcal{A}$ has an associated game potential $\mathcal{W}(u)$. If u maximizes $\mathcal{W}(u)$ locally in $(\mathcal{A}_G, d)$ it is a Nash equilibrium

Finding local maxima

Corollary 3.1

If $\mathcal{W}(u) \geq \mathcal{W}(u')$ for all $u' \in \Gamma_{\mathcal{A}}(u)$ then u is a Nash equilibrium.

Model Setup

Unfortunately $\mathcal{A}$ is not an easy set to understand or imagine. In this chapter I will show that the metric space is isometric (or quasiisometric) to a different space that is easier to imagine.

- Define an easier to imagine quasimetric space
- Associate $\mathcal{A}_G$ to a new space through a quasiisometry
- Translate potential function results to this new space by way of the quasiisometry

Subgraph metric space

Call the set of bridge free subgraphs of G, $\mathcal{SC}_G$. We say that two subgraphs of $\mathcal{SC}_G$ are adjacent in the quasimetric D if and only if they are separated by a Single Face Rewiring (SFR).

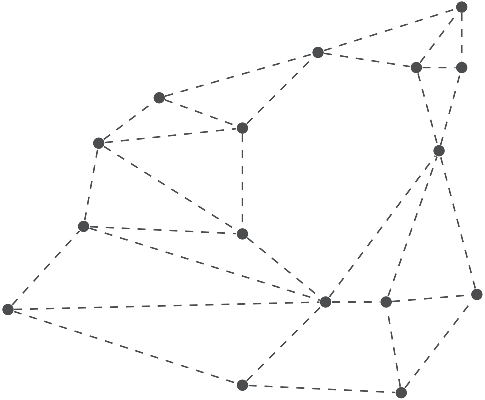
Single Face Rewiring (SFR)

For a simple planar graph G and a subgraph $s \in \mathcal{SC}_G$, a Single Face Rewiring from s to r is a resampling of edges from G so that, for one face f of G

1. **Locality:** Any edge not incident to the face f which was in s is still in r after the SFR.
2. **Closure:** Edges incident to f cannot be added in such a way that would create a bridge in r.
3. **Connectivity:** If two departure vertices were connected by a path along f in s but are not connected by a path along f in r, then they must be entirely disconnected in r.

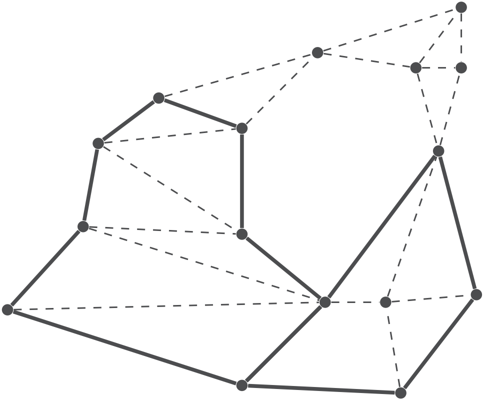
The bridge-free subgraph space

A planar graph G



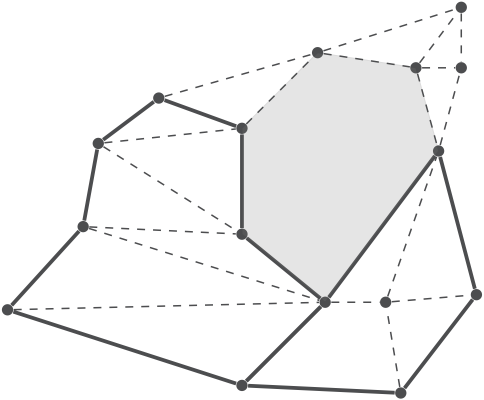
The bridge-free subgraph space

A bridge free subgraph $s \in \mathcal{SC}_G$



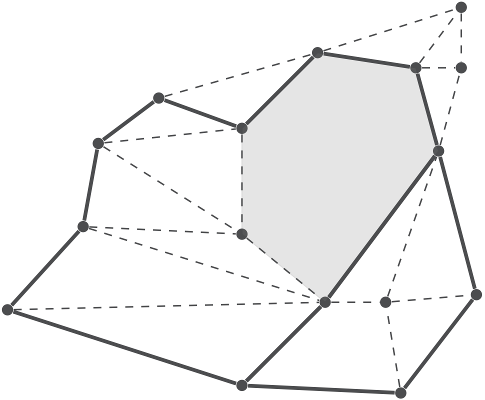
The bridge-free subgraph space

select a face f of G



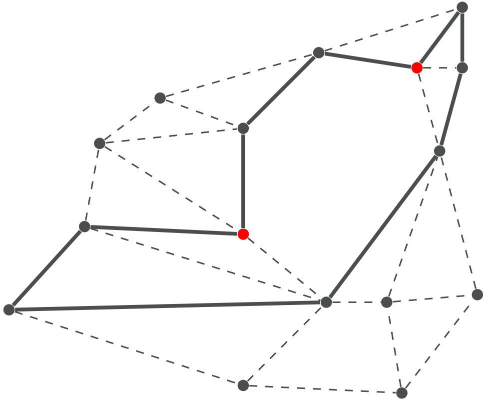
The bridge-free subgraph space

Resample edges of G incident to f to form new subgraph $s' \in \mathcal{SC}_G$



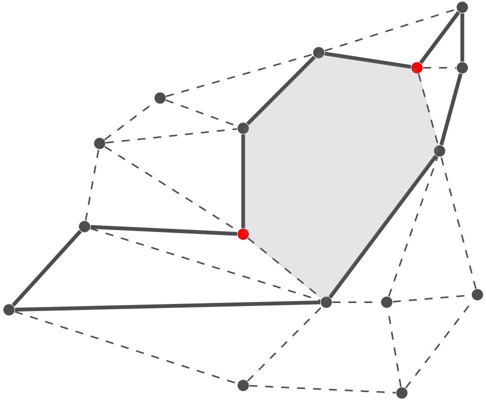
The bridge free subgraph space

another subgraph $s \in \mathcal{SC}_G$



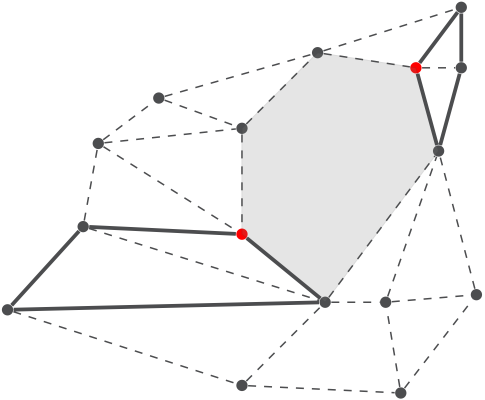
The bridge free subgraph space

If two departure vertices are initially connected by a path along f in s...



The bridge free subgraph space

... and are not connected by a path along f in s' they must be entirely discon-nected in s'

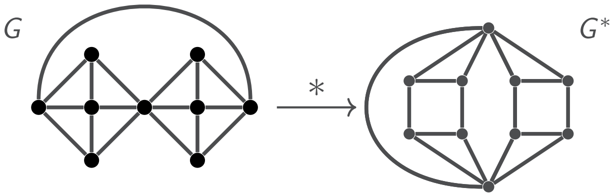


Graphical dual

Dual graph

Let G be a 3-edge connected planar graph the dual G^* is a simple planar graph.

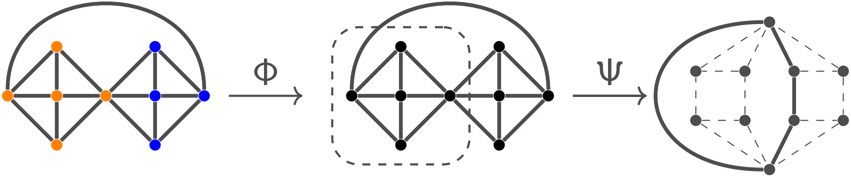
1. Every face in G is a vertex of G^*
2. An edge separating two faces in G corresponds to an edge between the two corresponding vertices in G^*
3. Thus, every vertex in G corresponds to a face of G^*.



Quasiisometry

$\Psi : \mathcal{Q}_G \to \mathcal{SC}_{G^*}$

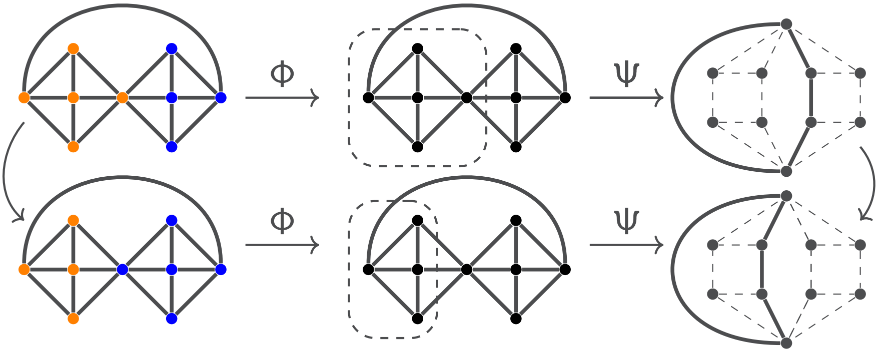
Let $\Psi : \mathcal{Q}_G \to \mathcal{SC}_{G^*}$ where if $e \in E(P)$ then $e^* \in \Psi(P)$. In this way $\Psi \circ \Phi u$ forms the **strategic boundaries** for the strategy profile u.



Quasiisometry

Lemma 3.6

For a planar 3-edge connected graph G, if $u \in \mathcal{A}_G$ has $\Psi u \in \mathcal{QC}_G$ and $d(u, u') = 1$ in $(\mathcal{A}_G, d)$ for some u', then $D(\Psi \circ \Phi u, \Psi \circ \Phi u') = 1$ in $\mathcal{SC}_{G^*}$



Proof of the quasiisometry

Assume that $d(u, u') = 1$ and that $\Phi u \in \mathcal{QC}_G$. We will show that there is a SFR between $s := \Psi \circ \Phi u$ and $s' := \Psi \circ \Phi u'$

Step 1: Locality

u and u' differ by a single strategic change, call the player that changed their strategy v and note that for every edge not incident to v, the players on either end of the edge did not change their strategy. If v corresponds to the face f in G^* consider an edge e^* not incident to f

$$e^* \in s \iff e \in E(\Phi u) \implies e \in E(\Phi u') \iff e^* \in s'$$

$$e^* \notin s \iff e \notin E(\Phi u) \implies e \notin E(\Phi u') \iff e^* \notin s'$$

Every edge of G^* not incident to f in s must also be present in s'

Proof of the quasiisometry

Assume that $d(u, u') = 1$ and that $\Phi u \in \mathcal{QC}_G$. We will show that there is a SFR between $s := \Psi \circ \Phi u$ and $s' := \Psi \circ \Phi u'$

Step 2: Closure

Closure can be proved entirely by the construction of Ψ.

Lemma 3.3

$$\Psi(P) \in \mathcal{SC}_{G*} \quad \forall P \in \mathcal{Q}_G$$

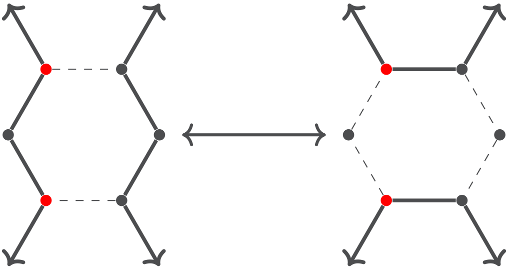
This is because a cut in G is isomorphic to a cycle in G^* and thus the cutset of a partition (the union of cuts) is isomorphic to the union of cycles in G^*. s and s' are therefore bridge free. (resampling the edges did not create a bridge in s').

Proof of the quasiisometry

Assume that $d(u, u') = 1$ and that $\Phi u \in \mathcal{QC}_G$. We will show that there is a SFR between $s := \Psi \circ \Phi u$ and $s' := \Psi \circ \Phi u'$

Step 3: Connectivity

Suppose there are two departure vertices a and b of f which are connected by a path incident to f in s but not connected by a path incident to f in s'. In order to be an SFR, a and b must be disconnected in s'

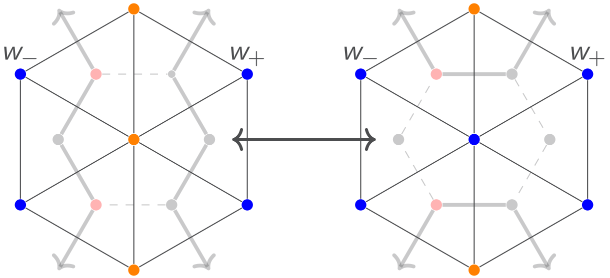


Proof of the quasiisometry

Assume that $d(u, u') = 1$ and that $\Phi u \in \mathcal{QC}_G$. We will show that there is a SFR between $s := \Psi \circ \Phi u$ and $s' := \Psi \circ \Phi u'$

Step 3: Connectivity

There must be at least two non adjacent edges incident to f in G^* not included in s' separating a and b. Call these missing edges e^*_+ and e^*_-. In G they correspond to edges connecting v to the vertices w_+ and w_- respectively. They must have been using the same strategy in u.



Proof of the quasiisometry

Assume that $d(u, u') = 1$ and that $\Phi u \in \mathcal{QC}_G$. We will show that there is a SFR between $s := \Psi \circ \Phi u$ and $s' := \Psi \circ \Phi u'$

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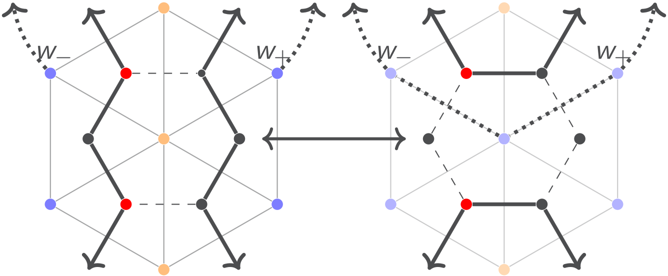
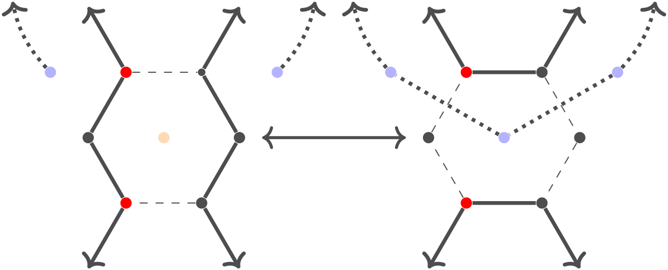


Proof of the quasiisometry

Assume that $d(u, u') = 1$ and that $\Phi u \in \mathcal{QC}_G$. We will show that there is a SFR between $s := \Psi \circ \Phi u$ and $s' := \Psi \circ \Phi u'$

Step 3: Connectivity

Therefore, there is a cycle starting and ending at v of players all using the same strategy in u'. This cycle defines a Jordan curve in the plane which separates a from b no edge of s' can cross the Jordan curve by construction so a and b are disconnected



Equilibrium result

Lemma 3.6

For a planar 3-edge connected graph G, if $u \in \mathcal{A}_G$ has $\Psi u \in \mathcal{QC}_G$ and $d(u, u') = 1$ in $\mathcal{A}_G$ for some u', then $D(\Psi \circ \Phi u, \Psi \circ \Phi u') = 1$ in $\mathcal{SC}_{G^*}$

Equilibrium result

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Theorem 3.2

If $s \in \mathcal{SC}_{G^*}$ satisfies

$$size(s) \leq size(r) \quad \forall r \in \Gamma_{G^*}(s)$$

where $\Gamma_{G^*}(s) := \{r \in \mathcal{SC}_{G^*}; D(s, r) \leq 1\}$ then there is a $u \in \Phi^{-1} \circ \Psi^{-1} s$ so that u is a Nash equilibrium

Equilibrium result

Lemma 3.6

For a planar 3-edge connected graph G, if $u \in \mathcal{A}_G$ has $\Psi u \in \mathcal{QC}_G$ and $d(u, u') = 1$ in $\mathcal{A}_G$ for some u', then $D(\Psi \circ \Phi u, \Psi \circ \Phi u') = 1$ in $\mathcal{SC}_{G^*}$

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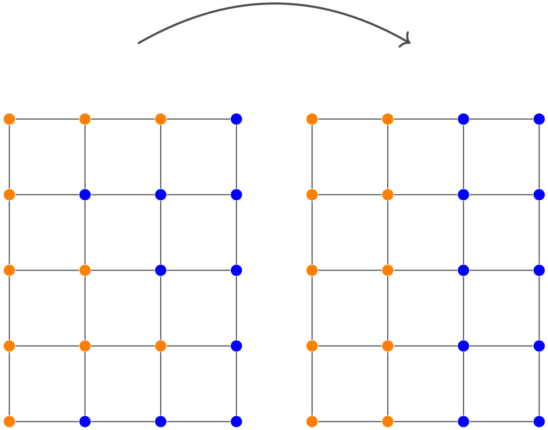
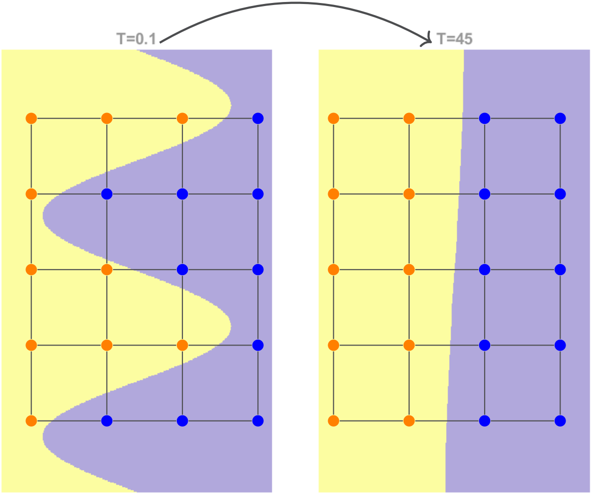
- When we restrict to two strategies MBR is equivalent to a "cycle shortening flow" (Thm 3.5)
- In a continuous player space MBR recapitulates mean curvature flow in the Cauchy setting (Thm 3.6)

Boundary minimization



Boundary minimization



Results of the dual model

	Simulation	Dual Model
Main Achievement	Observation	
Numerical Speed	✓	
Analytical Tractability	X	
Model Accuracy	✓ ✓	

Results of the dual model

	Simulation	Dual Model
Main Achievement	Observation	Intuition
Numerical Speed	✓	
Analytical Tractability	X	
Model Accuracy	✓ ✓	

Results of the dual model

	Simulation	Dual Model
Main Achievement	Observation	Intuition
Numerical Speed	✓	X
Analytical Tractability	X	
Model Accuracy	✓ ✓	

Results of the dual model

	Simulation	Dual Model
Main Achievement	Observation	Intuition
Numerical Speed	✓	X
Analytical Tractability	X	✓
Model Accuracy	✓ ✓	

Results of the dual model

	Simulation	Dual Model
Main Achievement	Observation	Intuition
Numerical Speed	✓	X
Analytical Tractability	X	✓
Model Accuracy	✓ ✓	✓

Results of the dual model

	Simulation	Dual Model
Main Achievement	Observation	Intuition
Numerical Speed	✓	X
Analytical Tractability	X	✓
Model Accuracy	✓ ✓	✓ ✓

Results of the dual model

	Simulation	Dual Model
Main Achievement	Observation	Intuition
Numerical Speed	✓	X
Analytical Tractability	X	✓
Model Accuracy	✓ ✓	✓ ✓

Main Result

The strategic boundaries minimize their lengths (area) in a parabolic, diffusive manner. Large regions of consensus are unlikely to change.

Replicator dynamics

Replicator Equation

$$\frac{d}{dt}p_i = p_i(f_i(p) - \varphi(p))$$

Replicator dynamics

Replicator Equation

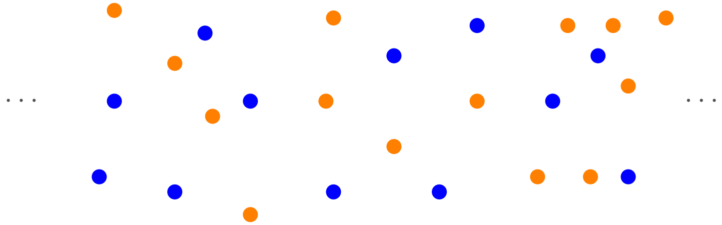
$$\frac{d}{dt}p_i = p_i(f_i(p) - \varphi(p))$$

In an infinite well mixed population on individuals playing pure strategies, p is a distribution over pure strategies $\{p_1, ..., p_m\}$, $f_i(p)$ is the fitness of playing i against p and $\varphi(p)$ is the average fitness of a player in p.

Replicator dynamics

Replicator Equation

$$\frac{d}{dt}p_i = p_i(f_i(p) - \varphi(p))$$



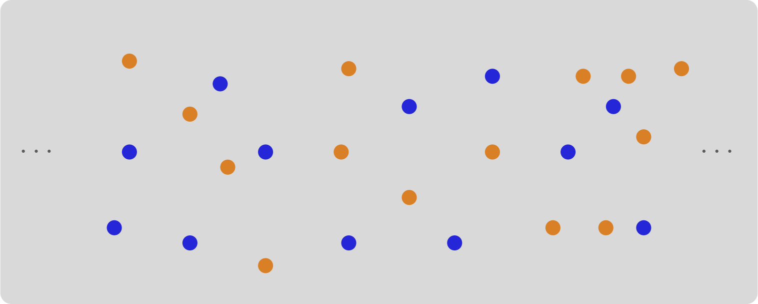
Original Setting

$$V = \mathbb{N} \qquad C = \{[1,0],[0,1]\} \subset \mathbb{R}^m \qquad p \in \Delta^{m-1}$$

Replicator dynamics

Replicator Equation

$$\frac{d}{dt}p_i = p_i(f_i(p) - \varphi(p))$$



Modifying the model to add structure

Consider this “infinite population” as a single patch

Replicator dynamics

Replicator Equation

$$\frac{d}{dt}p_i = p_i(f_i(p) - \varphi(p))$$

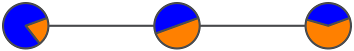
Modifying the model to add structure

Now for a single player, $u_v \in C := \Delta^{m-1}$

Replicator dynamics

Replicator Equation

$$\frac{d}{dt}p_i = p_i(f_i(p) - \varphi(p))$$



Modifying the model to add structure

$$f_v^i(u) = \sum_{w \in V} W_{w,v} \langle \hat{e}^i, Au_w \rangle \qquad \varphi_v(u) = \sum_{w \in V} W_{w,v} \langle u_v, Au_w \rangle$$

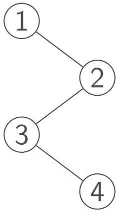
Structured replicator dynamics

A strategy profile is $u : V \to \Delta^{m-1}$ which is identical to $u \in (\Delta^{m-1})^V$. Thus we have an $n \cdot m$ dimensional system of ODEs

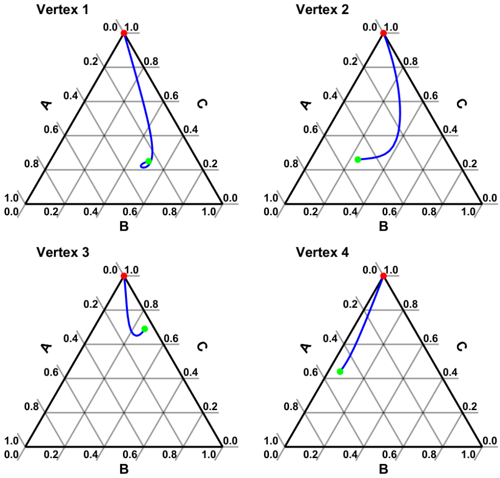
ODE System

$$\frac{d}{dt} u_v^i = u_v^i \langle \hat{e}^i - u_v, \sum_{w \in V} A u_v \rangle$$

Examples



$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Dynamic results

How do strategy profiles change in this model compared to the discrete time model?

Lemma 4.3

If $u(t)$ solves the IVP with $u(0) \in (\Delta^{m-1})^V$ then

$$\frac{d}{dt}\bigg|_{\zeta=0} w_v(u_v + \zeta \frac{d}{dt} u_v | u) \geq 0$$

with equality only when $\frac{d}{dt} u_v = \vec{0}$. This means the replicator dynamics represent a *Better Response* strategy revision protocol.

Better response

The fact that this is a “better response” strategy revision protocol means

- transient behavior of the ODE system may not exactly reflect the behavior of the discrete time game
- basins of stability are not necessarily the same across the two models

However, we can examine equilibria and their stability.

Theorem 4.1

If $u(t)$ is a convergent trajectory in the interior of $(\Delta^{m-1})^V$ then $\lim_{t\to\infty} u(t)$ exists and is a Nash equilibrium.

Persistence of mixed strategies

Stability results from ODE theory allow us to understand more about coordinating systems. for instance:

> *A strategy profile where adjacent players are playing overlapping mixed strategies is always unstable.*

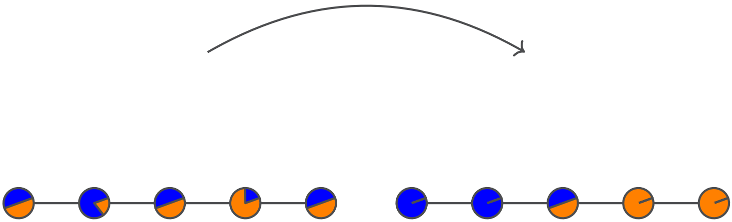
Lemma 4.9

In a pure coordination game, if an equilibrium strategy profile u^* has two players v and w satisfying

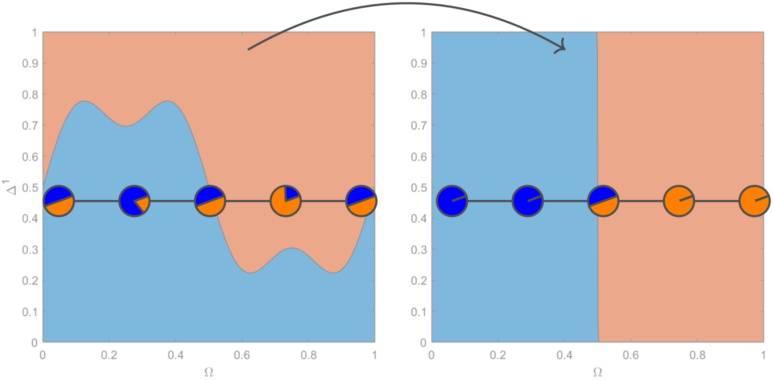
> There exists $i \in BR_v(u^*) \cap BR_w(u^*)$ such that either $u_v^i > 0$ and $u_w^i > 0$ or $u_v^i = 0$ and $u_w^i < 1$
>
> $|BR_v(u^*)| > 1$ and $|BR_w(u^*)| > 1$

then u^* is unstable.

Discretization



Discretization



Results of the ODE model

	Simulation	Dual Model	ODE Model
Main Achievement	Observation	Intuition	
Numerical Speed	✓	X	
Analytical Tractability	X	✓	
Model Accuracy	✓ ✓	✓ ✓	

Results of the ODE model

	Simulation	Dual Model	ODE Model
Main Achievement	Observation	Intuition	Stability
Numerical Speed	✓	X	
Analytical Tractability	X	✓	
Model Accuracy	✓ ✓	✓ ✓	

Results of the ODE model

	Simulation	Dual Model	ODE Model
Main Achievement	Observation	Intuition	Stability
Numerical Speed	✓	X	✓
Analytical Tractability	X	✓	
Model Accuracy	✓ ✓	✓ ✓	

Results of the ODE model

	Simulation	Dual Model	ODE Model
Main Achievement	Observation	Intuition	Stability
Numerical Speed	✓	X	✓
Analytical Tractability	X	✓	✓
Model Accuracy	✓ ✓	✓ ✓	

Results of the ODE model

	Simulation	Dual Model	ODE Model
Main Achievement	Observation	Intuition	Stability
Numerical Speed	✓	X	✓
Analytical Tractability	X	✓	✓
Model Accuracy	✓ ✓	✓ ✓	X

Results of the ODE model

	Simulation	Dual Model	ODE Model
Main Achievement	Observation	Intuition	Stability
Numerical Speed	✓	X	✓
Analytical Tractability	X	✓	✓
Model Accuracy	✓ ✓	✓ ✓	X ✓

Results of the ODE model

	Simulation	Dual Model	ODE Model
Main Achievement	Observation	Intuition	Stability
Numerical Speed	✓	X	✓
Analytical Tractability	X	✓	✓
Model Accuracy	✓ ✓	✓ ✓	X ✓

Main Result

Strategic gradients are unstable. Regions with high strategic gradients separate and regions with low strategic gradients, near pure strategies, diffuse.

Nonlocal diffusion

Comparable Strategies

What if two strategies, which are not identical, can still offer a positive payoff by being considered "close enough"

Nonlocal diffusion

Comparable Strategies

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Nonlocal diffusion

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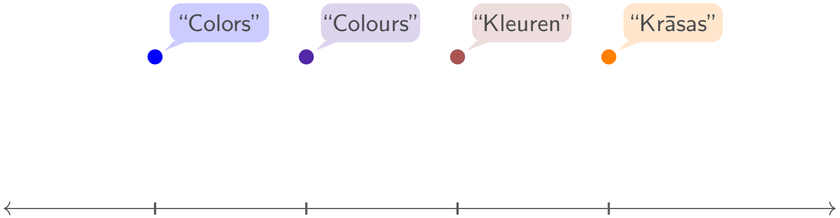

Nonlocal diffusion

Comparable Strategies

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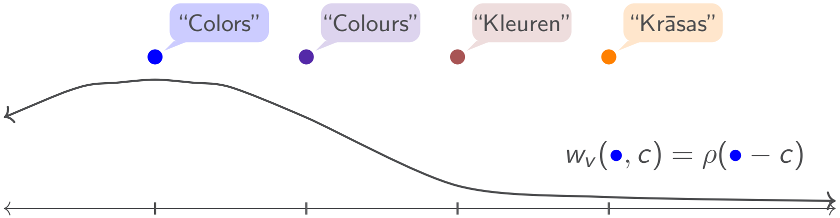

Continuous strategy space

Discrete Strategy Space

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \rightarrow \quad A' = \begin{bmatrix} 0.9 & 0.05 & 0 \\ 0.05 & 0.9 & 0.05 \\ 0 & 0.05 & 0.9 \end{bmatrix}$$

In this case, the payoff from playing pure strategies can be determined simply be the "distance" between those strategies

Continuous strategy space

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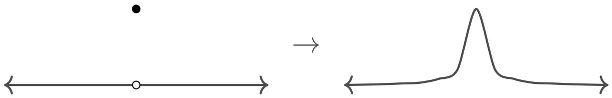
In this case, the payoff from playing pure strategies can be determined simply be the "distance" between those strategies

Continuous Strategy Space

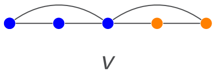


$\rho(z)$ is the payoff from playing a strategy c against strategy c' so that $c - c' = z$

Continuous player space

Discrete Player Space

Players V are vertices in the graph G



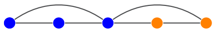
Payoffs are computed as a sum over all vertices weighted by $W_{v,w}$

$$\sum_{w \in V} W_{w,v} \rho(u_v - u_w)$$

Continuous player space

Discrete Player Space

Players V are vertices in the graph G



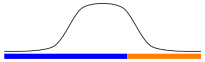
v

Payoffs are computed as a sum over all vertices weighted by $W_{v,w}$

$$\sum_{w \in V} W_{w,v} \rho(u_v - u_w)$$

Continuous Player Space

Players x are points in $\Omega \subset \mathbb{R}^n$



x

Payoffs are computed as in integral over Ω weighted by $K(x, y)$.

$$\int_\Omega K(x, y)\rho(u(x) - u(y))dy$$

Strategy revision protocol

We will be using the Myopic Best Response strategy revision protocol, but with modification

Proposition 5.1

Under myopic best response, bounded strategy profiles of the game with players $\Omega \subseteq \mathbb{R}^n$ choosing strategies in $\mathbb{R}$ will evolve as

$$\frac{\partial}{\partial t}u(x,t) = g[u](x,t) := \int_\Omega K(x,y)\rho'(u(x,t) - u(y,t))dy$$

so long as the following three hypotheses are met

(H1) Players change their strategies in arbitrarily small time steps Δt

(H2) Players incur a quadratic cost $\frac{h^2}{\Delta t}$ for changing their strategy

(H3) $\rho \in C^{1,1}(\mathbb{R})$ and $\rho(z) \leq Cz^2 + A$ for some nonnegative C, A

Existence, uniqueness, regularity

Theorem 5.1

The initial value problem $u_t = g[u]$ in Ω_τ has a unique continuous and bounded solution in Ω_τ for some time τ when $u(x, 0) = u_0 \in C_b^0(\Omega)$

Existence, uniqueness, regularity

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Theorem 5.2

If ρ satisfies $z \cdot \rho'(z) \leq 0$ (The coordination condition), then the initial value problem $u_t = g[u]$ in $\Omega \times \mathbb{R}_+$ has a unique continuous and bounded solution in $\Omega \times \mathbb{R}_+$ when $u(x,0) = u_0 \in C_b^0(\Omega)$.

Existence, uniqueness, regularity

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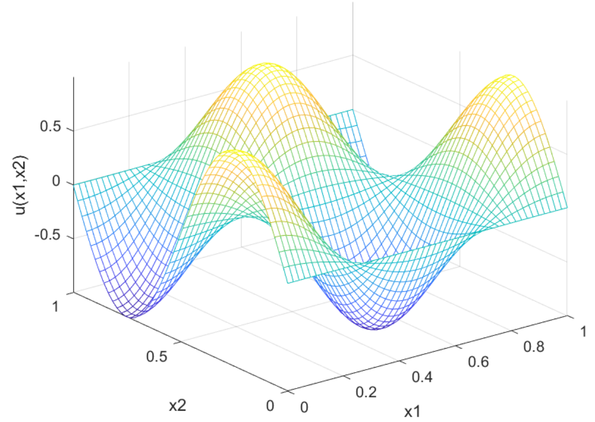
If ρ satisfies $z \cdot \rho'(z) \leq 0$ (The coordination condition), then the initial value problem $u_t = g[u]$ in $\Omega \times \mathbb{R}_+$ has a unique continuous and bounded solution in $\Omega \times \mathbb{R}_+$ when $u(x,0) = u_0 \in C_b^0(\Omega)$.

Theorem 5.3

If ρ satisfies the coordination condition and $u_t = g[u]$ in Ω_T and $u_0 \in C_b^{0,1}(\Omega)$ then $u \in W^{1,\infty}(\Omega_T)$ with

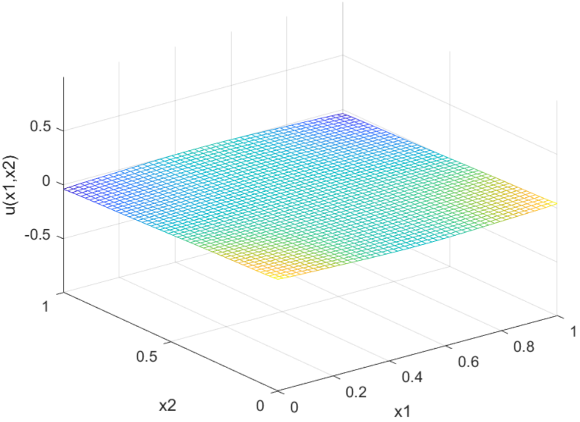
$$\|Du\|_{L^\infty(\Omega)} \leq (L_0 + CT)e^{cT}$$

Numerical simulations



The nonlocal diffusion equation is highly sensitive to changes in the recognition function

Numerical simulations



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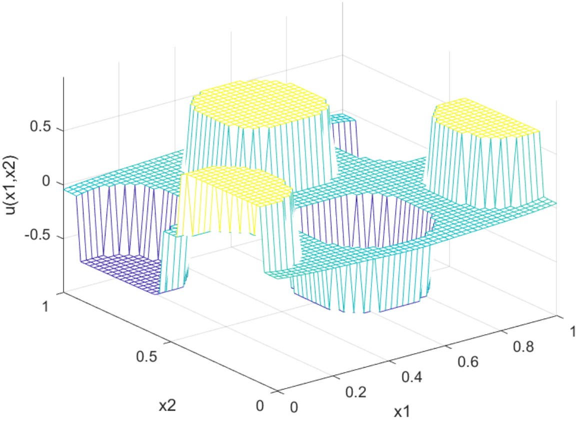
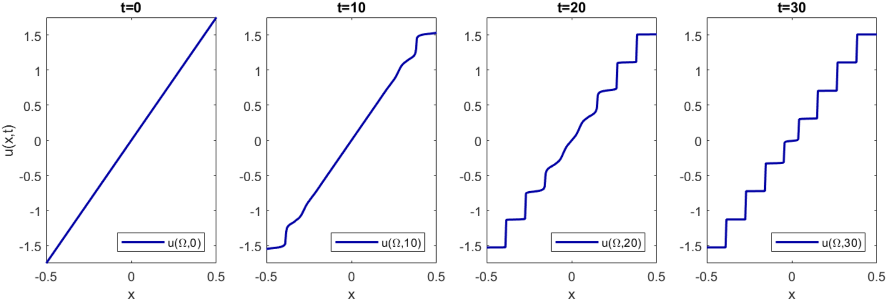


The nonlocal diffusion equation is highly sensitive to changes in the recognition function

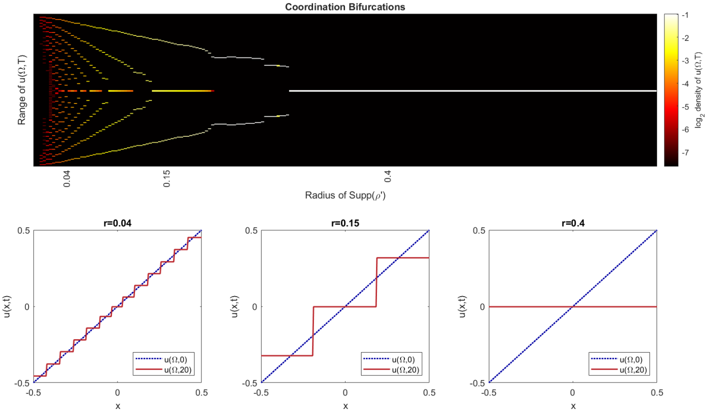
Discretization



Whenever $supp(\rho') \subset [-a, a]$ with only one zero in the interior of its support, then we always observe discretization

Discretization

The gaps between the points in the image of the limit are spaced in a predicable way which is related to the recognition function.



Discretization

The gaps between the points in the image of the limit are spaced in a predicable way which is related to the recognition function.

Theorem 5.6

Let Ω be a bounded domain in $\mathbb{R}^n$ and $K(x, y) \in C_b^0(\Omega; L^1(\Omega))$ be bounded below by λ, so that $\Omega \subset supp(K(x, \cdot))$. Let ρ' satisfy the coordination condition and have support $(-a, a) \subset \mathbb{R}$ with only one zero on the interior of its support. Under these conditions, if u satisfies $g[u] = 0$ in Ω and u is bounded then the image $u(\Omega)$ is a finite set of points separated by at least $|a|$ except possible at a set of measure 0.

Results of the nonlocal model

	Simulation	Dual Model	ODE Model	Nonlocal Model
Main Achievement	Observation	Intuition	Stability	
Numerical Speed	✓	X	✓	
Analytical Tractability	X	✓	✓	
Model Accuracy	✓ ✓	✓ ✓	X ✓	

Results of the nonlocal model

	Simulation	Dual Model	ODE Model	Nonlocal Model
Main Achievement	Observation	Intuition	Stability	Extension
Numerical Speed	✓	X	✓	
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Model Accuracy	✓ ✓	✓ ✓	X ✓	

Results of the nonlocal model

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Results of the nonlocal model

	Simulation	Dual Model	ODE Model	Nonlocal Model
Main Achievement	Observation	Intuition	Stability	Extension
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Analytical Tractability	X	✓	✓	✓
Model Accuracy	✓ ✓	✓ ✓	X ✓	

Results of the nonlocal model

	Simulation	Dual Model	ODE Model	Nonlocal Model
Main Achievement	Observation	Intuition	Stability	Extension
Numerical Speed	✓	X	✓	✓
Analytical Tractability	X	✓	✓	✓
Model Accuracy	✓ ✓	✓ ✓	X ✓	✓

Results of the nonlocal model

	Simulation	Dual Model	ODE Model	Nonlocal Model
Main Achievement	Observation	Intuition	Stability	Extension
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Results of the nonlocal model

	Simulation	Dual Model	ODE Model	Nonlocal Model
Main Achievement	Observation	Intuition	Stability	Extension
Numerical Speed	✓	X	✓	✓
Analytical Tractability	X	✓	✓	✓
Model Accuracy	✓ ✓	✓ ✓	X ✓	✓ X

Main Result

When strategies are comparable in a coordination game, strategy profiles act diffusively near consensus but will form discontinuities determined by the radius or recognition.

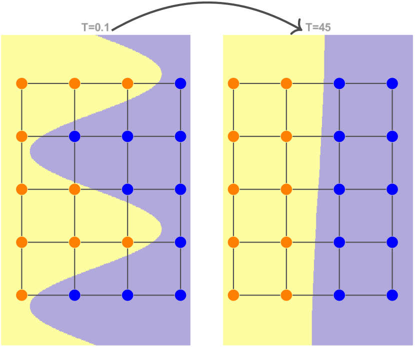
Model Comparison

	Simulation	Dual Model	ODE Model	Nonlocal Model
Main Achievement	Observation	Intuition	Stability	Extension
Numerical Speed	✓	X	✓	✓
Analytical Tractability	X	✓	✓	✓
Model Accuracy	✓ ✓	✓ ✓	X ✓	✓ X

Themes of coordinating systems

Diffusion Near Consensus

Models of coordinating systems have a parabolic nature, minimizing an "energy" related strategic gradients



In the dual model, strategic boundaries move in a diffusive parabolic manner

Themes of coordinating systems

Diffusion Near Consensus

Models of coordinating systems have a parabolic nature, minimizing an "energy" related strategic gradients

Near Consensus the Replicator model can be written as

$$\frac{d}{dt}u_v^i(t) = u_v^i \cdot \mathcal{L}u^i + (u_v^i)^2 - \langle u_v, g_v \rangle$$

where $\mathcal{L}$ is the graphical Laplacian. Although it is nonlinear, it shares similarities with the graphical diffusion equation.

Themes of coordinating systems

Diffusion Near Consensus

Models of coordinating systems have a parabolic nature, minimizing an "energy" related strategic gradients

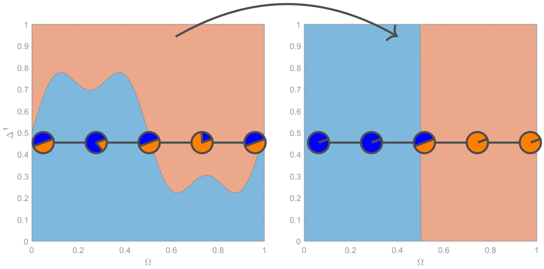
The nonlocal nonlinear diffusion model is obviously diffusive near consensus, when $\omega_u(\Omega) < \delta$ then $\rho'(z) \approx -cz$. This means

$$\int_\Omega K(x,y)\rho'(u(x,t)-u(y,t))dy \approx c \underbrace{\int K(x,y)(u(y,t)-u(x,t))dy}_{=\Delta_K u}$$

Themes of coordinating systems

Discretization at a Distance

Models of coordinating systems tend to form discontinuities near regions with large strategic gradients

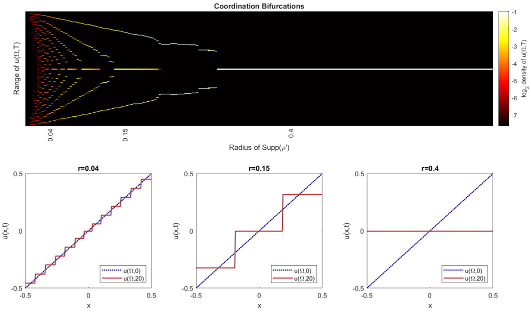


large strategic gradients collapse to discontinuities

Themes of coordinating systems

Discretization at a Distance

Models of coordinating systems tend to form discontinuities near regions with large strategic gradients

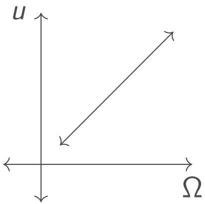


Discontinuities form with predictable regularity when strategies are comparable

Key insights

Strategic gradient collapse

A strategic gradient is, in general, unstable. It can collapse in two mutually exclusive ways.



Characteristics of the path along which the strategic/payoff gradient occurs, determine which type of collapse is promoted.

Key insights

Strategic gradient collapse

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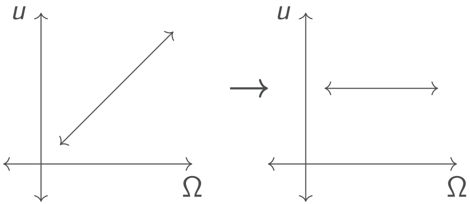

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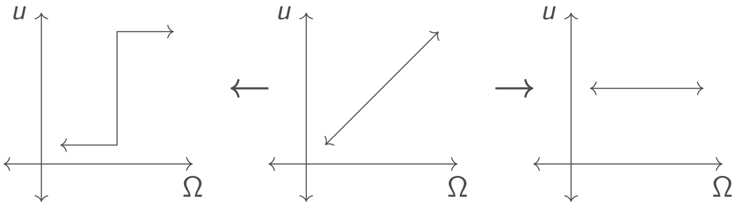

Key insights

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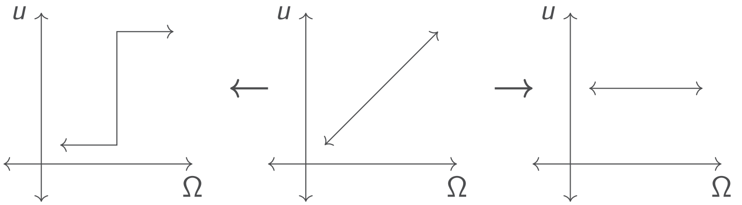
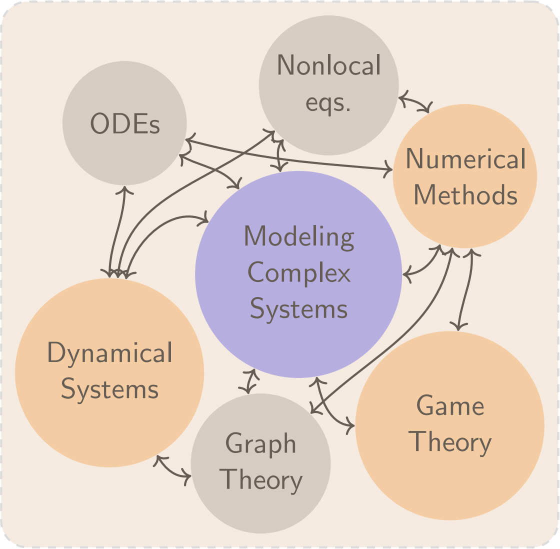


Characteristics of the path along which the strategic/payoff gradient occurs, determine which type of collapse is promoted.

Key insights



Coordinating systems are diffusive locally and discretizing globally

Determine path characteristics which promote gradient collapse in the player space.

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- NSF Division of Research on Learning: Developing an Early Understanding of Contagion in preschool- and kindergarten-aged Children

NSF

APPEX
Analysis and Prediction of Pandemic Expansion

NIMBioS
National Institute for Modeling Biological Systems

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Thank you to my Friends and Family

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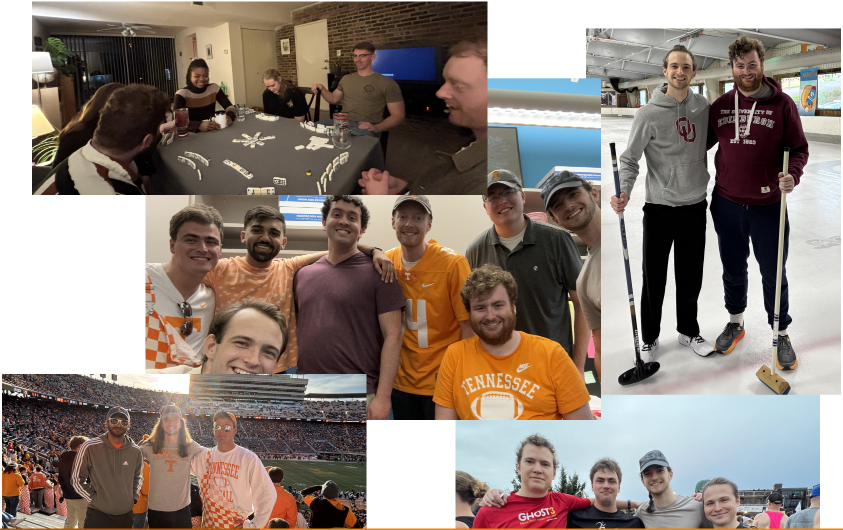

Future plans

PRINCETON UNIVERSITY

Levin Lab

Post-Doctoral Position

I am excited to be joining the Levin lab as a post doctoral research associate at Princeton University

Thank you

Questions?

For publications and code, visit my website

jmcalis.github.io