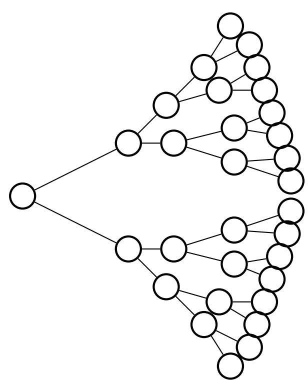
University of Tennessee - Knoxville

Mathematical Reasoning

Lecture Notes for Math 113

John McAlister



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Preliminaries

Welcome to math 113. This course is a survey of mathematics so it will probably be a course very different from the other math courses you've taken. It's an opportunity to get a survey of the math topics that you may not have encountered previously. The truth about math is that the math you see in high school is probably the least interesting (although the most helpful) subset of math we could pick. Hopefully in this survey of mathematics you'll see that there is more to it than equations and calculations.

Chapter 1

Sequence

1.1 Introduction to Sequences

A **sequence** is an ordered list of numbers. They can be finite or infinite. They can be defined in several ways. One way to define them is simply to write it out

$$\{1, 2, 3, 4, 5\}, \quad \{a_1, a_2, a_3, a_4\} \tag{1.1}$$

$$\{1, 1, 2, 3, 5, 8, 13, ...\} \tag{1.2}$$

When we write out a sequence like $\{a_1, a_2, a_3...\}$ the little subscript is called the **index** and it tells us where in the sequence that term falls. For instance a_4 is the fourth term in the sequence. a_{87} is the 87th term in the sequence.

We can try to write all our sequences like this but if we want to write down a very long sequence we need to use a "..." as in equation 2. If the pattern in the sequence is not very obvious the "..." is not very clear so we need some other way to write down the sequence. Another way to define them is to write a **general** expression (or an nth term expression) for each term. This is a way of writing down the nth term which only depends on the index n.

$$a_n = 2n + 1 \tag{1.3}$$

This formulation is helpful because if we have a general expression for a sequence and we want to know what the millionth term is, we can calculate it without having to find every previous term.

The last way to define a sequence is what is called a **recurrence relation**. This is a way of writing down the nth term which only depends on the previous term or terms. For this form, we need to be provided with some **initial condition** (we will always use a_1)

$$a_n = 2 + a_{n-1} \quad a_1 = 1 \tag{1.4}$$

Notice the difference between the general expression and the recurrence relation. The general expression depends on n and the recurrence relation depends on a_{n-1}

Our goals in this chapter will be to be able to look at a sequence and recognize what comes next, to translate between different ways of writing a sequence, and to become familiar with important sequences.

1.2 Constant sequences

The most basic sequence is the constant sequence

$$\{2, 2, 2, 2, 2, 2, ...\} \tag{1.5}$$

This sequence has the general expression

$$a_n = 2 \tag{1.6}$$

and the recurrence relation

$$a_n = a_{n-1} \quad a_1 = 2 \tag{1.7}$$

Importantly, every term in this sequence is the same.

1.3 Arithmetic sequences

In an arithmetic sequence, we add some number to the previous term. It is helpful to start with the recurrence relation here

$$a_n = a_{n-1} + 2 \quad a_1 = 1 \tag{1.8}$$

This means that the first term will be 1 and that we will calculate the nth term by looking at the previous term "a_{n-1}" and adding 2. We call the "+2" the "**common difference**". If we write out the first couple terms we can see that this is

$$\{1, 3, 5, 7, 9, ...\} \tag{1.9}$$

It takes some thinking but we can write a general expression for this sequence First consider $a_2 = a_1 + 2$ so $a_3 = (a_1 + 2) + 2 = a_1 + 2 \cdot 2$. Furthermore, $a_4 = (a_1 + 2 \cdot 2) + 2 = a_1 + 3 \cdot 2$. We can now recognize the pattern

$$a_n = a_1 + 2(n - 1) \tag{1.10}$$

For a general common difference $a_n = d + a_{n-1}$ then we can say $a_n = d(n - 1) + a_1$

1.3.1 Example 1

An example is

$$a_n = a_{n-1} + 4 \quad a_1 = 0 \tag{1.11}$$

In this case, our sequence starts at zero and our common difference is 4. Thus we can write is as $a_n = 4(n - 1)$

1.3.2 Example 2

If the common difference is negative it might look like this

$$a_n = a_{n-1} - 3 \quad a_1 = 2 \tag{1.12}$$

The sequence is $\{2, -1, -4, -7, -10...\}$ and the nth term expression is $a_n = -3(n-1) + 2$

1.4 Geometric sequences

Whereas, in the arithmetic sequence we added to each term, in the geometric sequence we multiply to get the next term. For example:

$$a_n = 2 * a_{n-1} \quad a_1 = 1 \tag{1.13}$$

Again we can write out the first couple terms

$$\{1, 2, 4, 8, 16, 32, ...\} \tag{1.14}$$

To write the general solution we do the same thing as before, observe $a_2 = a_1 \cdot 2$ and $a_3 = (a_1 \cdot 2) \cdot 2 = a_1 2^2$ so $a_4 = (a_1 2^2) \cdot 2 = a_1 2^3$. Thus we see the pattern emerge

$$a_n = a_1 2^{(n-1)} \tag{1.15}$$

In the above example we multiply by two each time but we can multiply by any number, this number is called the **common ratio** which we call r. In general is we have the sequence $a_n = r a_{n-1}$ then $a_n = a_1 r^{n-1}$

1.4.1 Example 1

As an example if $a_n = 3a_{n-1}$ and $a_1 = 1$ then $a_n = 3^{n-1}$.

1.4.2 Remark

There are some special cases. If $a_1 = 0$ then, regardless of the common ratio, every term will be zero and the sequence will be constant. If $r = 1$ then again we have a constant sequence because our recursion relation can just be written as $a_n = a_{n-1}$

These sequences, arithmetic and geometric appear often in computer science and biology in addition to many other disciplines.

1.5 Triangular, Square, and Tetrahedral numbers

There are many sequences for which we cannot easily write down a recurrence relation or a nth term expression. An example of this could be the triangle numbers.

Imagine you are trying to stack cups. You may start with a line of 4 cups on the table, on those you can easily stack three cups. On top of the three cups you can place two more and on top of those you can place 1. The total number of cups you have used is 10. This configuration is very natural because 10 is a triangle number. If you wanted to go a layer higher you would have to use 15 cups. This is the next triangle number.

Formally the nth triangle number is the sum of the first n counting numbers.

$$\begin{aligned} t_1 &= 1 \\ t_2 &= 3 = 1+2 \\ t_3 &= 6 = 1+2+3 \\ t_4 &= 10 = 1+2+3+4 \\ &\vdots \\ t_n &= 1+2+3+...+n \end{aligned} \tag{1.16}$$

A general rule for this sequence is possible and to talk about it we need to discuss a huge name in mathematics, Gauss. Carl Friedrich Gauss was a great mathematician from a very early age but he was also, reportedly, an obnoxious students. One time, he got in trouble with a teacher and, as a punishment, the teacher had him add the first 100 counting numbers. We know that this sum of the first 100 counting numbers is the 100th triangle number. The teacher assumed that this would take hours because he would have to add 100 numbers, but Gauss found a shortcut. His argument went like this:

$$t_{100} = 1+2+...+100 \tag{1.17}$$

He then he doubled it and writes it in a particular way.

$$\begin{aligned} t_{100} + t_{100} = &\ 1+2+3+...+99+100 \\ &+100+99+98+...+2+1 \end{aligned} \tag{1.18}$$

If we add each column together we can rewrite this as

$$2 \cdot t_{100} = 101+101+101+...+101+101 \tag{1.19}$$

On the right hand side of the equation we add (101) 100 times so we can write

$$2t_{100} = 101 \cdot 100 \tag{1.20}$$

Thus Gauss showed that

$$t_{100} = \frac{101 \cdot 100}{2} = 5050 \tag{1.21}$$

1.5.1 Theorem

The general rule for any triangle number is

$$t_n = \frac{n(n+1)}{2} \tag{1.22}$$

1.5.2 Proof

We can go through the previous calculation carefully using a general "n" instead of 100.

The **square numbers** are another sequence like the triangle numbers which have their basis in geometry. If you are arranging cubes into squares the smallest square you can make uses just 1 cube, the next used 4 cubes, the next uses 6 and so on. We are familiar with these numbers as the square numbers. Their sequence goes as

$$\{1, 4, 9, 16, 25...\} \tag{1.23}$$

If we return to the stacking cups example we might want to make a triangular pyramid, a **tetrahedron**, we would start by making the first layer as a triangle, then the next layer would be a triangle smaller than the first until we get to the top of the pyramid where we use just one cup. The number of cups we use to build such a pyramid is a tetrahedral number. Notice that, because we are adding triangular layers, the nth tetrahedral number will be the sum of the first n triangle numbers.

The first smallest tetrahedron we can make is just a single cup. the next biggest has a layer with 3 cups and a layer with 1 cup so we use 4 cups total. Next we would have to start with a 6 cup layer, then a 3 cup layer, than a 1 cup layer. Thus the next tetrahedral number is 10. We can see the pattern below

$$\begin{aligned}
&T_1 = 1 = t_1 \\
&T_2 = 4 = t_1 + t_2 \\
&T_3 = 10 = t_1 + t_2 + t_3 \\
&T_4 = 20 = t_1 + t_2 + t_3 + t_4 \\
&\quad \vdots \\
&T_n = t_1 + t_2 + t_3 + ... + t_n
\end{aligned} \tag{1.24}$$

1.6 Pascal's Triangle

Some sequences are not lists of numbers like they ones we have encountered already. On such sequence is called Pascal's triangle named after Blaise Pascal. Pascal's triangle is found by a simple rule: an entry is the sum of the two entries on either side of it in the row above. If there is no entry on one side or the other, it is counted as a zero. This is a confusing rule so let us compute the first few rows together. Imagine a pyramid

$$\begin{array}{c}
\square \\
\square \quad \square \\
\square \quad \square \quad \square \\
\square \quad \square \quad \square \quad \square
\end{array} \tag{1.25}$$

We will put a 1 in the first box. This is our initial condition.

$$\begin{array}{c} 1 \\ \square\ \square \\ \square\ \square\ \square \\ \square\ \square\ \square\ \square \end{array} \tag{1.26}$$

In the next row we add the number above and to the left and the number above and to the right of each empty box to get their entry. If there is no number in that spot we count it as zero. So for the first entry, the number above to the left is a imaginary 0 and the number above and two the right is a 1 so we put a 1 in that box. The same can be found for the second entry in that row

$$\begin{array}{c} 1 \\ 1\ \ 1 \\ \square\ \square\ \square \\ \square\ \square\ \square\ \square \end{array} \tag{1.27}$$

The next row we continue with this same process. The first entry is a $0+1=1$, as before, the second entry is $1+1=2$, and the last entry is a $1+0=1$.

$$\begin{array}{c} 1 \\ 1\ \ 1 \\ 1\ \ 2\ \ 1 \\ \square\ \square\ \square\ \square \end{array} \tag{1.28}$$

Continue this in the fourth row we can see the first entry is $0+1=1$, the next is $1+2=3$ the next is $2+1=3$ and the last is $1+0=1$.

$$\begin{array}{c} 1 \\ 1\ \ 1 \\ 1\ \ 2\ \ 1 \\ 1\ \ 3\ \ 3\ \ 1 \end{array} \tag{1.29}$$

We can continue this pattern forever to get Pascal's triangle. It is more than a pretty doodle, it is actually closely related with probability which we will talk about late in this course, and some topics in algebra and combinatorics which we will not encounter here.

We also see many of the sequences we have already discussed in this section present in the triangle. along diagonals we can see a constant sequence, an arithmetic sequence, the triangle numbers and the tetrahedral numbers.

1.7 Fibonacci Sequences

The last sequence we will discuss and perhaps the most famous is the Fibonacci Sequence named for the Italian mathematician Fibonacci. It is given by a simple recurrence relation but we cannot easily write down a general expression for the sequence. The recurrence relation and the initial conditions are

$$F_n = F_{n-1} + F_{n-2} \quad F_1 = 1, F_2 = 1 \tag{1.30}$$

This seems like a rule so simple that it can't be practical but the Fibonacci sequence appears everywhere in nature. From swirling plants to the shapes of shells, the Fibonacci sequence seems to be intimately related to the natural world.

1.7.1 Example 1

Using this recurrence relation, we can write out the first 10 terms of the Fibonacci sequence.

$$\{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...\} \tag{1.31}$$

Notice that if we chose different initial conditions we would get a different sequence

1.7.2 Example 2

If we had the Fibonacci type sequence: $F_n = F_{n-1}+F_{n-2}$ with initial conditions $F_1 = 2$, $F_2 = -3$ Then we get the sequence

$$\{2, -3, -1, -4, -5, -9, -14, -23, -37, ...\} \tag{1.32}$$

Something similarly connected to the natural world is what is called the **golden ratio**. It is also sometimes called the golden proportion or the divine proportion. To define the ratio we consider a rectangle whose long side is a and short side is b. This rectangle is a **golden rectangle** if the ratio a/b is the same as the ratio $\frac{b}{a-b}$. (Figure 1)

In math with use the letter φ to talk about the golden ratio. It has a value of $\varphi \approx 1.618$ This is the ratio of height to width of many ancient structures and is considered very aesthetically pleasing. Additionally it can be found in shapes through out nature from spiraling shells to spiraling galaxies.

These things may not seems related but they indeed are. After a very long time, the ratio of on Fibonacci number to the next tends towards φ

$$\frac{F_{n+1}}{F_n} \approx \varphi \quad \text{For large n} \tag{1.33}$$

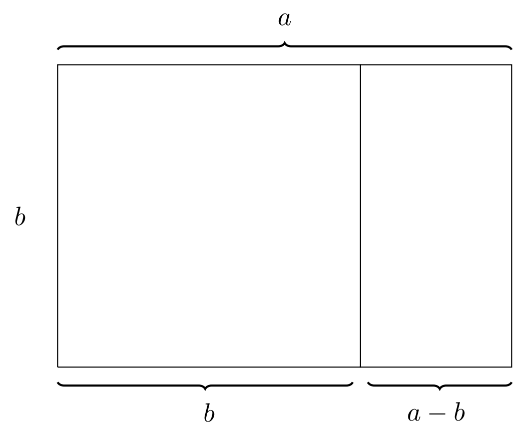


Figure 1.1: golden rectangle (not to scale)

1.7.3 Derivation

We can find the value of the golden ratio using the rectangle in figure 1. Let $b = 1$ and we will solve for a Because of the properties of the rectangle we know that $a = \frac{1}{a-1}$. We can rewrite this as $a^2 - a - 1 = 0$ If we use the quadratic equation we solve for a as $a = \frac{1+\sqrt{5}}{2} = \varphi$ (we disregard the negative solution). There is a similarly interesting derivation to show the relationship between φ and the Fibonacci sequence but it's far beyond the scope of the course so I won't include it here. You should ask me about it though.

1.8 Modeling with Sequences

Sequences are an extremely helpful tool when it comes to modeling. Mathematical modeling is the practice of describing an observable phenomenon in a simplified numerical way in order to draw general conclusions or make predictions about the modeled system. When we seek to model something which changes in time, sequences can be a really powerful tool. One of the most famous examples is infectious disease.

1.8.1 Example 1

Suppose you are an epidemiologist seeking to understand how a certain infectious disease will spread. It is hard to make a long term predictions right away but through some research it has been concluded that if one person is infected, after 1 week they will pass that infection on to three people. Hidden in this information is a sequence. Let a_n be the number of infected individuals at week n. Suppose that in week 1 there was 1 infected individual $a_1 = 1$. We know that for every infected individual in one week, there will be 3 infected individuals the next week. This gives us a recurrence relationship

$$a_n = 3 \cdot a_{n-1} \quad a_1 = 1 \tag{1.34}$$

We call this a mathematical model of epidemiological spread. Now if you want to predict how many infected people there will be in 6 weeks, you can turn this recurrence relation into a general expression

$$a_n = 3^{n-1} \tag{1.35}$$

and solve $a_6 = 3^5$. We can see from the general expression that as n increases so will the number of infected individuals

1.8.2 Example 2

If instead we had the same set up but research had found that every infected individual infects 0.5 people on average each week and there was wide spread infection initially ($a_n = 100$) we can use the same tools to make predictions about this system

$$a_n = 0.5 \cdot a_{n-1} \quad a_1 = 100 \iff a_n = 100 \cdot 0.5^{n-1} \tag{1.36}$$

Now we see that as n gets large, a_n will continue to decrease.

What we have seen in these past two examples is how sequences are used by applied mathematicians and scientists as a way of describing and understanding the world. By learning more analysis of sequences (which mathematical biologists call difference equations) we can make more and more sophisticated models to describe the world around us!

Chapter 2

Number Theory

2.1 Sets

Before we talk in depth about Number theory, it will be helpful to talk about **Sets**. Sets are just collections of numbers and they do not have a prescribed order. Like sequences they can be finite or infinite and we have lots of ways to write them down. In this class we won't think much about writing down infinite sets other than the really important ones which already have names. There are, however, some facts about sets that are helpful to know.

If you are given a set, a **subset** is a set that contains some of the elements of the original set. For instance take the set $S = \{1, 2, 3, 4, 5, 6, 7, ...\}$. This is a set we will come to know as the natural numbers. A subset of that set would be $O = \{1, 3, 5, 7, 9, ...\}$. These are the odd numbers. We write that one set is a subset of another as $O \subset S$.

The concept of subsets will be important as we talk about types of numbers but also as we talk graph theory, probability, and logic so make sure you're comfortable with this definition.

2.2 Types of Numbers

We break up our number system into several types of numbers. The ones that we are most familiar with are the **Counting numbers**. There are also called the **Natural numbers**

$$\mathbb{N} = \{1, 2, 3, 4, 5, 6...\} \tag{2.1}$$

These are the numbers we use to index our sequences but of course they are not the only numbers. They are not even the only whole numbers. **Whole numbers** are considered the counting numbers and zero so

$$\omega = \{0, 1, 2, 3, 4, 5, 6, ...\} \tag{2.2}$$

The whole numbers are not used for counting in most disciplines instead they are used to answer the question "How many?". The natural numbers are a subset of the whole numbers so every natural number is a whole number but there is a whole number which is not a natural number.

These are, of course, still more numbers. **Integers** are positive or negative whole numbers. so

$$\mathbb{Z} = \{... - 3, -2, -1, 0, 1, 2, 3, ...\} \tag{2.3}$$

These are the sets of numbers we consider when we talk about number theory. It is important to note that the whole numbers are a subset of the integers. Observe that all counting numbers are whole numbers and all whole numbers are integers thus every counting number is an integer.

Rational Numbers are the set of all numbers which can be written as $\frac{a}{b}$ where a and b are integers.

$$\mathbb{Q} = \left\{\frac{a}{b}; a, b \in \mathbf{Z}, b \neq 0\right\} \tag{2.4}$$

2.2.1 Example 1

Think about if every integer is a rational number? The answer to this question is a resounding yes. We can observe this because any integer can be expressed as a fraction of itself divided by 1. Does this mean that the integers are a subset of the rationals? Yes it does!

$$\mathbb{Z} \subset \mathbb{Q} \tag{2.5}$$

There are numbers which we sometimes talk about that are not rational numbers and so they cannot be represented as a fraction. These numbers are called **irrational numbers**. They are just the numbers that are "left over" if we ignore the rational numbers.

Finally, all the rational and irrational numbers together make up what are called the **real numbers** $\mathbb{R}$. The real numbers are all the numbers between $-\infty$ and ∞ on the number line. There are numbers which are not real but we will not consider these in this class.

2.2.2 Example 2

Knowing these classifications allows us to ask questions about numbers, for instance. If I take two *integers* and multiply them together, can I ever get a number that is not an integer? To answer this we think about what multiplication is, if we imagine that there are a groups of b items then it's clear that there are no partial or fractional items so we get a whole number.

2.2.3 Example 3

Another question I could ask is, If a and b are both natural numbers, is $a-b$ also a natural number? To answer this question I can find a **Counter Example**. A counter example is simply an example which disproves a claim. So if my claim is that $a-b$ is a natural number, a counter example would be that $2-8=-6$ which is not a counting number.

2.2.4 Example 4

One of my favorite such questions is, If we multiply a rational number by a rational number, will it still be rational?

To solve this problem we remember that rational numbers can be expressed as fractions of integers so we could represent one rational number as $\frac{c}{d}$ and the other as $\frac{f}{g}$ the product of these two is $\frac{cf}{dg}$. Now notice that the product of two integers is an integer so this new number, is a fraction if integers. Thus it is a rational number.

2.2.5 Remark

What we have just shown is a property called **Closure**. If we add two numbers from a set, and their sum is still in that same set, we say that that set is "closed to addition" Likewise is we multiply two numbers from a set and their product is still in that same set, we say that that set is "closed to multiplication."

Another interesting question we might ask about these types of numbers is "How large is this set?" The answer seems to be "infinitely large" for all of the sets so they are all somehow the same size. However we can show that these sets of numbers are not all the same size, indeed they have different **cardinality**. If we have time in class to talk about this we'll go through it because it's very cool but you won't be assessed on it.

2.3 Prime Numbers

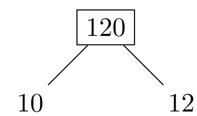
An important set of numbers that we have not talked about yet is the prime numbers. These are natural numbers greater than 1 which can not be divided evenly by any other whole number greater than 1. Another way to say that is "a number is **Prime** if it can only be evenly divided by 1 and itself". (As a reminder, if an integer divides another integer evenly, it means that their quotient is an integer.)

Prime numbers build up every natural number. To investigate this we use what is called the **prime factorization**. The prime factorization is a way of writing a number as the product of primes so

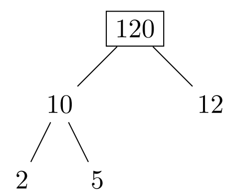
$$\begin{aligned} 2 &= 2 \\ 3 &= 3 \\ 4 &= 2 \cdot 2 \\ 5 &= 5 \\ 6 &= 2 \cdot 3 \\ 7 &= 7 \\ 8 &= 2 \cdot 2 \cdot 2 \\ 9 &= 3 \cdot 3 \\ 10 &= 2 \cdot 5 \end{aligned} \tag{2.6}$$

2.3.1 Example 1

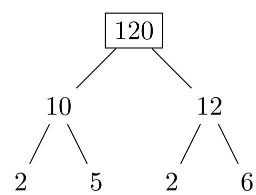
We can find this prime factorization by what is called a **factorization tree**. If we start with a number we are interested in factoring. for instance: 120, we can keep dividing it into smaller and smaller whole numbers like so First I notice that $120 = 10 \cdot 12$



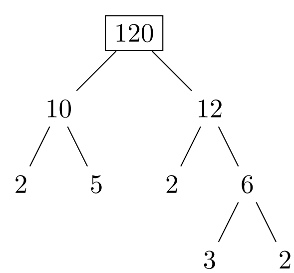
Then I notice that $10 = 2 \cdot 5$ so I can further break this down as



Now we notice that $12 = 6 \cdot 2$ so we can break this down further as



Lastly, 6 can be broken into $6 = 2 \cdot 3$ so we complete our factorization as



Now notice that all of the numbers at the bottom of the tree are prime, they cannot be broken down any further. Thus we have shown that $120 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5$

2.3.2 Theorem 1

There are a couple important results regarding prime numbers. The first is called the **Fundamental Theorem of Arithmetic**. This tells us that every natural number has a unique prime factorization. The proof is beyond the scope of this course.

2.3.3 Theorem 2

The other important result is the **Infinite Primes Theorem**. It says that there are an infinite number of prime numbers.

2.3.4 Exercise

Because there are an infinite number of primes it would be great if we could look at a number and use a rule to determine if it is prime. unfortunately (or fortunately) no such rule exists. There are schemes by which we can find primes. The most popular is called the **Sieve of Eratosthenes**

Lets list all the natural numbers in order. This is a tall order to write on this paper so I will write the natural numbers between 1 and 30.

$$\begin{array}{cccccccccc} & 02 & 03 & 04 & 05 & 06 & 07 & 08 & 09 & 10 \\ 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 & 29 & 30... \end{array} \tag{2.7}$$

The sieve works in the following way. We start by marking the number 2 then crossing out every other number which is a multiple of two.

$$\begin{array}{cccccccccc} & \underline{02} & 03 & \cancel{04} & 05 & \cancel{06} & 07 & \cancel{08} & 09 & \cancel{10} \\ 11 & \cancel{12} & 13 & \cancel{14} & 15 & \cancel{16} & 17 & \cancel{18} & 19 & \cancel{20} \\ 21 & \cancel{22} & 23 & \cancel{24} & 25 & \cancel{26} & 27 & \cancel{28} & 29 & \cancel{30}... \end{array} \tag{2.8}$$

Now, at every step here after, we underline the the next number which has not been marked yet, here it is three, and cross out every multiple of that number. After the second step our sieve looks like this

$$\begin{array}{cccccccccc} & \underline{02} & \underline{03} & \cancel{04} & 05 & \cancel{06} & 07 & \cancel{08} & \cancel{09} & \cancel{10} \\ 11 & \cancel{12} & 13 & \cancel{14} & \cancel{15} & \cancel{16} & 17 & \cancel{18} & 19 & \cancel{20} \\ \cancel{21} & \cancel{22} & 23 & \cancel{24} & 25 & \cancel{26} & \cancel{27} & \cancel{28} & 29 & \cancel{30}... \end{array} \tag{2.9}$$

We continue this same process indefinitely and what remains is all the primes, every other number is crossed out.

$$\begin{array}{cccccccccc} & \underline{02} & \underline{03} & \cancel{04} & \underline{05} & \cancel{06} & \underline{07} & \cancel{08} & \cancel{09} & \cancel{10} \\ \underline{11} & \cancel{12} & \underline{13} & \cancel{14} & \cancel{15} & \cancel{16} & \underline{17} & \cancel{18} & \underline{19} & \cancel{20} \\ \cancel{21} & \cancel{22} & \underline{23} & \cancel{24} & \cancel{25} & \cancel{26} & \cancel{27} & \cancel{28} & \underline{29} & \cancel{30}... \end{array} \tag{2.10}$$

This unfortunately cannot be done forever so while it is a theoretical way to arrive at the complete list of primes, it will take forever so we cannot use it. Thus there are still many questions about primes that we cannot answer

2.3.5 conjecture 1

For instance, the **Twin Prime Conjecture** has to do with so called **Twin Primes**. Twin primes are prime numbers which are only different by two. For instance, 5 and 7 are twin primes, 11 and 13 are twin primes, 17 and 19 are twin primes and so on. There is a conjecture that states that there are an infinite number of twin primes. This has yet to be proven and is an open area of research in number theory. If you can prove the twin prime conjecture I will give you an A for the class.

2.3.6 conjecture 2

Another important conjecture is the **Goldbach conjecture** which posits that any even number greater than 2 is the sum of two prime numbers. We can confirm this easily for all the primes we've identified above, $4 = 2+2$, $6 = 3+3$, $8 = 3+5$ and so on. If you can find an even number which is not the sum of two primes, I will give you an A for the class.

2.4 GCD and LCM

One important use for prime factorization is the **Greatest common divisor** and the **Least common multiple**. The Greatest common divisor (GCD) or two natural numbers is the largest natural number which evenly divides the two. We find the GCD by examining the prime factorization. If we want ot find the GCD or 12 and 16 then we observe that $12 = 2 \cdot 2 \cdot 3$ and $16 = 2 \cdot 2 \cdot 2 \cdot 2$. The Greatest common divisor will be the product of all the prime factors the two numbers have in common. Here both have 2 factors of 2 so the GCD is $2 \cdot 2 = 4$. We write this as

$$GCD(12, 16) = 4 \tag{2.11}$$

2.4.1 Example 1

As another example consider $GCD(27, 18)$. To solve this observe that $27 = 3 \cdot 3 \cdot 3$ and $18 = 3 \cdot 3 \cdot 2$. We see that both have two factors of 3 so their GCD is $3 \cdot 3 = 9$.

2.4.2 Example 2

As a final example consider $GCD(13, 9)$. 13 is itself prime so that is its only prime factor. $9 = 3 \cdot 3$ so there are no prime factors in common. Because of this, the greatest number which divides both of these numbers evenly is 1 so

$$GCD(13, 9) = 1 \tag{2.12}$$

The Least common multiple of two numbers is the smallest number which is a multiple of both numbers. We can again use the prime factorization to find the LCM. The LCM will be all of of the prime factors, the maximum number of times they are expressed in the prime factorization. This is a confusing explanation so an example will be helpful. To find the LCM of 12 and 16, recall that $12 = 2 \cdot 2 \cdot 3$ and $16 = 2 \cdot 2 \cdot 2 \cdot 2$. Between the two factorization, 2 is present a maximum of four times and 3 is present a maximum of once. Thus the LCM is the product of $2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 = 48$.

2.4.3 Example 3

Another example would be $LCM(27, 18)$. Recall that $27 = 3{\cdot}3{\cdot}3$ and $18 = 3{\cdot}3{\cdot}2$. 2 is present a maximum of once, and 3 is present a maximum of three times so the LCM is the product $3 \cdot 3 \cdot 3 \cdot 2 = 54$.

2.4.4 Example 4

A final example is $LCM(18, 6)$ Observe that $18 = 3 \cdot 3 \cdot 3$ and $6 = 3 \cdot 2$ so2 is present at most once, and 3 is present is at most twice to the LCM is the product $2 \cdot 3 \cdot 3 = 18$. Therefore $LCM(18, 6) = 18$. Notice that if a is a multiple of b the $LCM(a, b) = a$.

2.4.5 Remark

An interesting fact that might help in computing LCM or GCD is the for any two natural numbers a, b, we know $GCD(a, b) \cdot LCM(a, b) = a \cdot b$. I recommend spending some time convincing yourself that it's true.

2.4.6 Example 5

The GCD and LCM have some helpful applications. One application involves problems of different patterns coinciding. For instance, suppose I work every other day and my favorite coworker only works every 7 days. If we both work today, how long will it be until we are on the schedule together again?

We solve this by noticing I work on days which are multiples of two $\{2, 4, 6, 8, ...\}$ and they work on days that are multiples of seven, $\{7, 14, 21, ...\}$ so to find a day we work together we want to find a common multiple. The least common multiples of 2 and 7 is 14 so we can say that we will work together again in 14 days

2.4.7 Example 6

Another similar example would be the fascinating life cycle of the cicada. There are many different species of cicada and most lay eggs in the ground. These eggs typically take years to hatch and emerge. If one species takes 8 years to emerge and another take 6 years to emerge, how often do they emerge on the same year?

Again to answer this question notice that one species emerges on years which are a multiple of 8 and the other emerges on years which are a multiple of 6 to the least common multiple is $LCM(8,6) = 24$. They will emerge together every 24 years

In general, cicadas want to avoid emerging on the same year as another species so interestingly most cicada species stay buried underground for prime numbers of years. Because of this their LCMs are very large and they do not frequently coincide.

Chapter 3

Problem Solving

3.1 Dimensional Analysis

Dimensional analysis is the use of **Conversion Factors** to express a value in different units or understand the conversion between two directly related quantities. A conversion factor is a ratio of units which is equal to one. This can be. This is best motivated by an example. If I am running a 5 kilometer race and I want to know how long I am running in miles, I will need to use a conversion factor. I know that there are 1.6 kilometers in a mile so I can write

$$1.6k = 1mi \tag{3.1}$$

Another way to write this equation is

$$\frac{1.6k}{1mi} = 1 \text{ or } \frac{1mi}{1.6k} = 1 \tag{3.2}$$

Both of these are ratios of units which are equal to one. These are what we call conversion factors. Now, if I seek to convert $5k$ into units of miles, I need to I need to find a way to re-express this number in different units *without changing the value.* The distance should not change when we convert units, only the way we express it. We know that if we multiply something by 1 we do not change the value so if we multiply $5k$ by 1 we will not change the length of the race. We can use a special form of 1, carefully selected, to change the units without changing the value. Let's use the special form of 1 that we found above

$$5k \cdot \frac{1mi}{1.6k} = 3.125\frac{\not{k}\,mi}{\not{k}} = 3.125mi \tag{3.3}$$

3.1.1 Example 1

Now, If I am running this race but I am running at an 8 minute mile pace, I might want to know how long it will take me to finish the race. This again is another example of dimensional analysis wherein we are converting between two directly related quantities. Here time and distance are directly related by the conversion factor

$$\frac{8min}{1mi} \tag{3.4}$$

Now to convert from distance in kilometers to time in minutes we can string these conversion factors together.

$$5k \cdot \frac{1mi}{1.6k} \cdot \frac{8min}{1mi} = \frac{5 \cdot 8\cancel{k}\,\cancel{mi}\,min}{1.6\cancel{k}\,\cancel{mi}} = 25min \tag{3.5}$$

Again we have not changed the "value" of the length of the race we have only changed the way we express the race from distance units of k to time units of min.

3.1.2 Example 2

If you know that one Gallon of water is 3785 mL, one mL of water is one g of water, and one pound on earth is 454 grams, how many pounds is a gallon of water? To answer this question we start with a gallon of water and try to work our way to pounds.

$$1G \cdot \frac{3785mL}{1G} \cdot \frac{1g}{1mL} \cdot \frac{1lb}{454g} = 8.34lb \tag{3.6}$$

The goal in many of these problems will be to string together conversion factors to get from one unit to another. Distance and time are not the only application of this method, weight, mass, flow rate and volume are all commonly used in problems like these.

3.2 Scientific Notation

We have, in these past sections, written down many numbers without issue. However, there are many numbers which are difficult to write. Namely the very large numbers and the very small numbers are impossible to write on one page without some other tool; the tool we introduce in this section is called scientific notation.

Scientific notation is a way or writing a number by the product of a number between 1 and 10 and power of 10. For instance here are some examples

$$\begin{aligned} 2456 &= 2.456 \times 10^3 \\ 0.00000045 &= 4.5 \times 10^{-7} \end{aligned} \tag{3.7}$$

To be able to work with scientific notation we need to know a few facts about exponents

$$\begin{aligned} x^a \times x^b &= a^{a+b} \\ \frac{x^a}{x^b} &= a^{a-b} \\ x^0 &= 1 \\ x^{-a} &= \frac{1}{x^a} \\ (x^a)^b &= x^{a \cdot b} \end{aligned} \tag{3.8}$$

Scientific notation is an important part of being conversant in math and science because often important mathematical or scientific results are reported in scientific notation because they are too big or too small to report.

Let us practice expressing a number in scientific notation. The main principle of scientific notation is that multiplying by 10 moves the decimal point to the right and dividing by 10 moves it to the left. Observe

$$345.6 = 34.56 \times 10 = 3.456 \times 10^2 \tag{3.9}$$

So if we simply count the number of places the decimal point needs to move in order to get a number between 1 and 10 we find the power of 10 in our final answer. For instance in the number 24000.00, the decimal point needs to move 4 places to the right in order to get to the number 2.40 which is between 1 and 10. This tells us we can write $24000.00 = 2.40 \times 10^4$. Practice on these examples

3.2.1 Examples 1-4

$$\begin{aligned} 0.000034 &= 3.4 \times 10^{-5} \\ 47800000 &= 4.78 \times 10^7 \\ 0.00000006 &= 6 \times 10^{-8} \\ 10000 &= 1 \times 10^4 \end{aligned} \tag{3.10}$$

So far we have only seen examples of numbers which I can type out. This is natural on account of the fact we are reading these lecture notes. This notation becomes really useful when we want to write something like $8.3 \times 10^{10^{100}}$. If I tried to type out this number I would run out of storage on my computer.

It is easy to lose scale when numbers are presented in scientific notation. As an example consider how long 1 second is verses 1×10^6 verses 1×10^9. A second we agree is a short length of time but 1^6 seconds is the same as 6.6 days and 1×10^9 is a bit more than 18 years. This time analogy can be helpful when considering the magnitude of number like one million (10^6) and one billion (10^9).

We will also need to practice the skill of converting from scientific notation to decimal notation. The conversion is exactly the opposite of its counterpart above. We noticed above that we need only count the places the decimal point needs to move to find the exponent. Now, given an exponent, we will simply move the decimal point that many places to the left or right. We will try it with these examples

3.2.2 examples 5-8

$$\begin{aligned} 4.2 \times 10^3 &= 4200 \\ 5.8 \times 10^{-2} &= 0.058 \\ 1 \times 10^5 &= 100000 \\ 8.322 \times 10^0 &= 8.322 \end{aligned} \tag{3.11}$$

Finally, it is occasionally important to do operations with numbers in scientific notation. One solution to do this quickly can always be converting to decimal notation and operating then converting back but this is not always necessary. To add and subtract numbers is scientific notation we can rewrite one number so that it has the same exponent as another. Then we can carry out the addition or subtraction and return it to correct scientific notation.

3.2.3 Example 9

$$4.21 \times 10^5 + 3.2 \times 10^4 \tag{3.12}$$

We notice that we can write $4.21 \times 10^5 = 4.21 \times 10 \times 10^4 = 42.1 \times 10^4$ so we rewrite our problem as

$$42.1 \times 10^4 + 3.2 \times 10^4 \tag{3.13}$$

When both of our numbers have the exponent on the 10, we can add or subtract them freely like

$$(42.1 + 3.2) \times 10^4 = 45.3 \times 10^4 \tag{3.14}$$

Now we need only convert back to correct scientific notation by moving the decimal point and adjusting the exponent accordingly

$$45.3 \times 10^4 = 4.52 \times 10^5 \tag{3.15}$$

Multiplication and division is even simpler and requires no conversion into decimal. Recall, by the rules of exponents listed above, when we multiply $10^a \cdot 10^b$ we get 10^{a+b}.

3.2.4 Example 10

$$3.2 \times 10^5 \cdot 1.6 \times 10^7 \tag{3.16}$$

We can rearrange this as

$$3.2 \cdot 1.6 \times 10^5 \cdot 10^7 \tag{3.17}$$

Now we can multiply the first two parts together without a problem. $3.2 \cdot 1.6 = 6.4$. The last two parts multiply together using the rule mentioned above. $10^5 \cdot 10^7 = 10^{12}$. This gives us our answer

$$3.2 \times 10^5 \cdot 1.6 \times 10^7 = 6.4 \times 10^{12} \tag{3.18}$$

Occasionally, after the multiplication the number will not be in correct scientific notation, the decimal point may need to be moved and the exponent may need to be adjusted accordingly.

Chapter 4

Geometry

4.1 Interior angle measures, areas, and volumes

Geometry may be familiar to you as the study of angles areas and volumes. All of these things are parts of geometry but should be thought of as the tools geometrists uses rather than the goal of the study. In this section we will learn some of these tools so that we can use them to answer more interesting questions.

4.1.1 Angles

The first tool we have is angle measure, Although you are familiar with what an **angle** is, we define it as two **Rays** (Lines going of infinitely in one direction) from a single **Vertex** (Fig. 4.1). The measure of an angle is the amount of rotation from one side to the other. Angles are measured in degrees. Below, observe some important angles to know (Fig 4.2).

Angles can range from $0°$ to $360°$. It is an interesting note that a $0°$ angle and a $360°$ angle are indistinguishable.

We will not, at this moment, do very much with angle measure. The study of angles, triangles, and their relationships is called trigonometry. Although trigonometry is very interesting, it takes more time than we have to get to the interesting stuff, so we will not discuss it in this class.

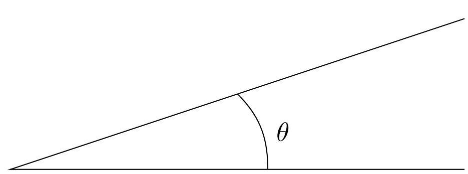


Figure 4.1: An angle with measure θ

Figure 4.2: 90° and 180° angles

4.1.2 Lines

Another way we measure geometric objects is with length. This one is quite familiar to us. We use length to measure lines. Lines or line segments are usually named by their endpoints as below (Fig 4.3)

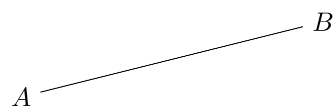


Figure 4.3: The line $\overline{AB}$

To talk about the length of the line $\overline{AB}$ I can say $AB = x$ to mean that $\overline{AB}$ is x units long.

4.1.3 Area

Length is a good descriptor for one dimension but for a figure in two dimensions we need a different measure. For this we use "area." In one dimension the shape we discussed was a line. In two dimensions we will discuss **Polygons**, which are closed figures in a plane determined by three or more straight line segments. Triangles and squares are common examples but there are an infinite number of polygons which do not have specific names (Fig 4.4).

As a side note, mathematicians have a nice way of naming polygons in general. The small polygons have names like triangle and rectangle but as the number of sides increase we will usually call them by the Greek word for the number of sides with the prefix "-gon" or, if the polygon has n sides we may also call it an "n-gon."

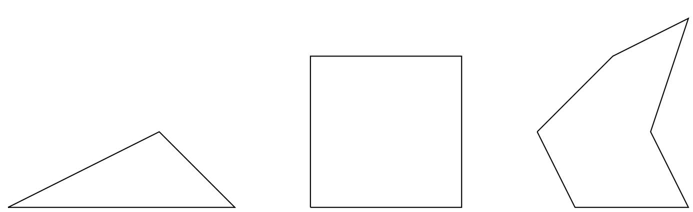


Figure 4.4: examples of polygons

3 sides	triangle
4 sides	quadrilateral
5 sides	pentagon
6 sides	hexagon
7 sides	heptagon
8 sides	octagon
$\vdots$	$\vdots$
35 sides	triacontakaipentagon
$\vdots$	$\vdots$
10^{10} sides	megagon
$\vdots$	$\vdots$
n sides	n-gon

Naming polygons is one thing, but measuring their area is not as easy. There is no general way to measure the area of a polygon, it often requires special formulas and always relies on the lengths of sides and measures of angles.

There are several polygons which we can easily take the area of.

4.1.4 Example 1

The rectangle, which is a quadrilateral which has only right angles, has area which is just the product of the perpendicular side lengths. We know this as "Length times width."

$$A = l \cdot w \tag{4.1}$$

4.1.5 Example 2

The triangle has an area which is given by one half of the product of the length of one side and the height of remaining vertex away from that side. This is more commonly known as "one half base times height"

$$A = \frac{1}{2} b \cdot h \tag{4.2}$$

4.1.6 Example 3

One important two dimensional shape that we've not listed yet is the circle. It is not a polygon because it does not have any straight lines or any corners. It is defined as "all of the points on a plane with are a distance r away from a center point." We know what a circle looks like. That distance r is called the radius. The area of a circle is given as πr^2 where pi is an irrational number close to 3.1415.

$$A = \pi r^2 \tag{4.3}$$

4.1.7 Volume

Just as we have length for one dimension and area for two dimensions. We can measure 3 dimensional figures by volume. Three dimensional figures have vertices (corners) edges (lines between corners) and faces (polygons made out of the edges). Many of the three dimensional figures we will care about are called **Polyhedrons**. These are defined as three dimensional figures which have faces that are all polygons. Again there is no general way to calculate volume.

These are the ways that we measure figures in one, two and three dimensions. There are other dimensions and other ways to measure figures but we will not discuss them here.

4.1.8 Example 4

Although nothing above is a very earth shattering concept, we can use it to solve some fun visual problems. In this section we will talk about problems with composite figures. A composite figure just means a figure which is made of many shapes. The simplest example is in figure 4.5. If $AB = 5$ and $CB = 2$, then what is AC? It is clear that $AC + CB = AB$ so we can solve for $AC = AB - CB = 5 - 2 = 3$. This is one of the important skills we will practice in this section. We have to think about how different small components of the figure come together to form the larger picture. If we know about each of the small parts we can learn something about the larger figure.

Figure 4.5: A simple composite figure.

4.1.9 Example 5

The same thing works in higher dimensions. For instance, consider the *inscribed square problem*(Fig 4.6). This problem will use the same mathematical reasoning skills as before. If we have a circle of radius 1 which fits perfectly in a square, what is the area of the square?

To solve this problem we think of the properties of a circle. Every point on the circle is 1 away from its center so $LO = 1$ We can add lengths together and say that $LR = 2$. Furthermore we can see that $AC = 2$ because $\overline{AC} = \overline{LR}$. This is a square (a type of rectangle whose length and width are the same) so $\overline{AB} = \overline{AC}$. Now we can say the area of this square is $AB \times AC = 2 \cdot 2 = 4$. Even with minimal information we can use the definitions of shapes like the circle and the square solve for areas of composite figures.

We can take the example a step further and ask “What is the area of the region outside of the circle but inside the square?”. If we take the area of the

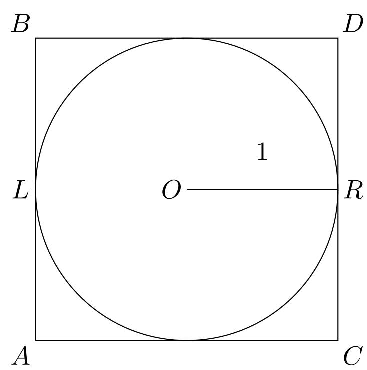


Figure 4.6: The inscribed square

square, which we just found to be 4 and subtract the area of the circle (π) we get the area of the region outside the circle but inside the square.

4.2 Similar figures

Now that we know some of the important measures of geometry, we can use them to solve problems involving so called Similar figures. We say that two figures are **similar** if and only if all of the corresponding angles have the same measure and the ratios of corresponding sides are the same. For example these two squares are similar (Fig 4.6).

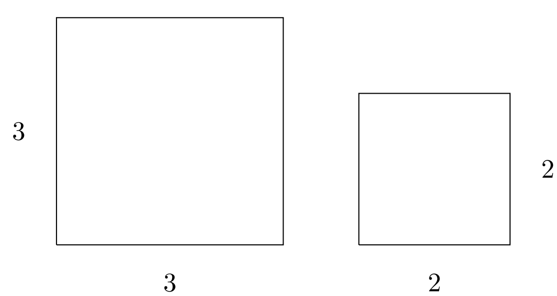


Figure 4.7: similar squares

We know that they are similar because every angle is has a measure of $90°$ and the ratios of all the side lengths are 1. This is an interesting piece of trivia itself but this definition allows us to ask two types of questions.

You may be given two figures and be asked to determine if they are similar or you may be told that two figures are similar and be asked to determine certain pieces of missing information about one or the other. We first consider the former.

To determine if two figures are similar we need to check to ensure that they satisfy the definition. This can be tricky because I can rotate or flip a figure around without changing the figure so finding a correspondence between angles and sides can be difficult lets look at an example.

4.2.1 Example 1

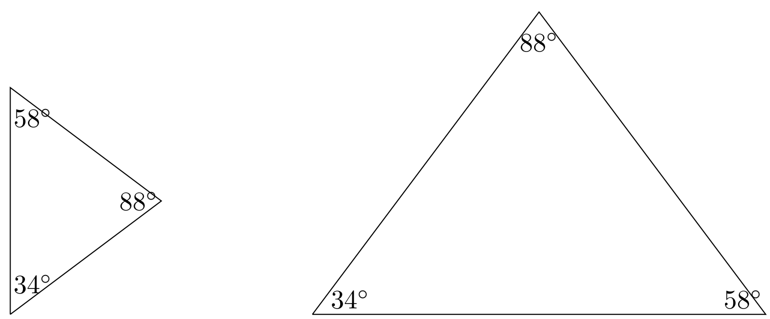


Figure 4.8: Are this figures similar?

Let us first only consider angle measures. We need to find a correspondence between angles in these two triangles. It may may be hard to identify by just looking at it so here are some helpful hints. We can see that some of these angles have the same measure, lets orient one of these angle pairs to line up.

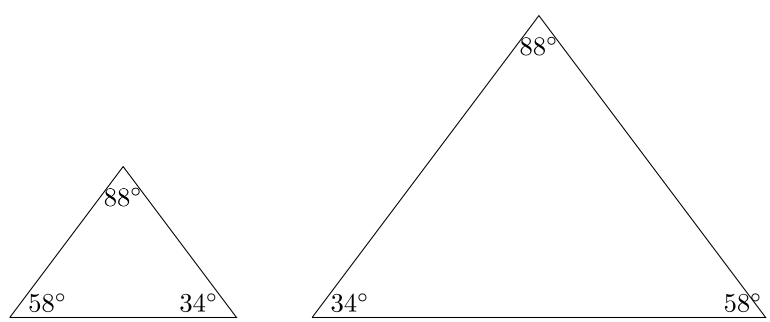


Figure 4.9: Are this figures similar after rotation?

Again we see that the angles measures of the angles are all there but they don't line up. We can get them to line up if we imagine we flip over the smaller triangle. From there we can determine if there is a correspondence between all of the angles.

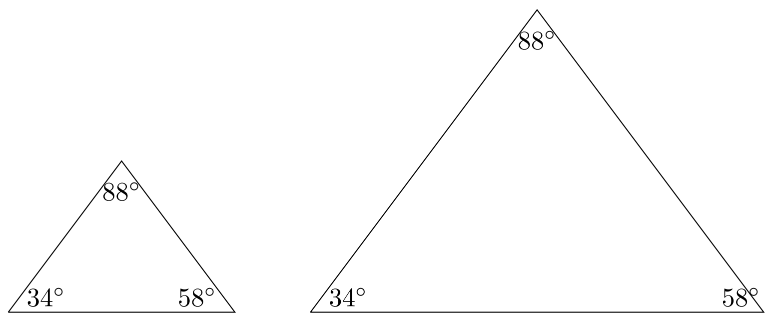


Figure 4.10: Are this figures similar after rotation and flipping?

It is very clear now that there is indeed a correspondence between all of the angles. In other words, every angle in the small triangle has the same measure as the angle in the same relative position in the big triangle. Now we can consider the question of side lengths. I'll redraw the figures now and label the points.

Recall to determine is the figures are similar the ratios of corresponding side lengths must be the same. I will show below that they are.

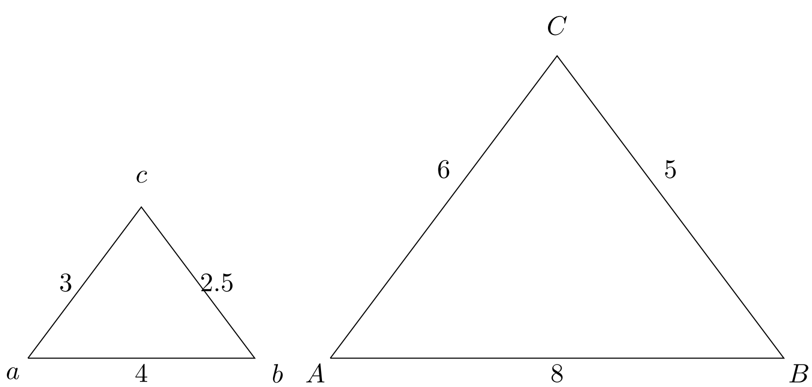


Figure 4.11: Are this figures similar after rotation and flipping?

$$\begin{aligned} \frac{ac}{ab} = \frac{3}{4} = \frac{6}{8} = \frac{AC}{AB} \\ \frac{bc}{ab} = \frac{2.5}{4} = \frac{5}{8} = \frac{BC}{AB} \\ \frac{ac}{bc} = \frac{3}{2.5} = \frac{6}{5} = \frac{AC}{BC} \end{aligned} \tag{4.4}$$

In equation 4.4 we see that corresponding ratios of side lengths are always the same between these two figures and so we know that they are similar. As a piece of math trivia, it's cool to notice that If two triangles have the same three angles, no matter the order they appear, they are necessarily similar.

This was an arduous task, and not one that is altogether that interesting. The other type of question relating to similar figures is more interesting and more directly applicable. If you know two figures are similar you can use the definition of similar figures to fill in missing information about one or both of these figures. Here's an example (Fig 4.12)

4.2.2 Example 2

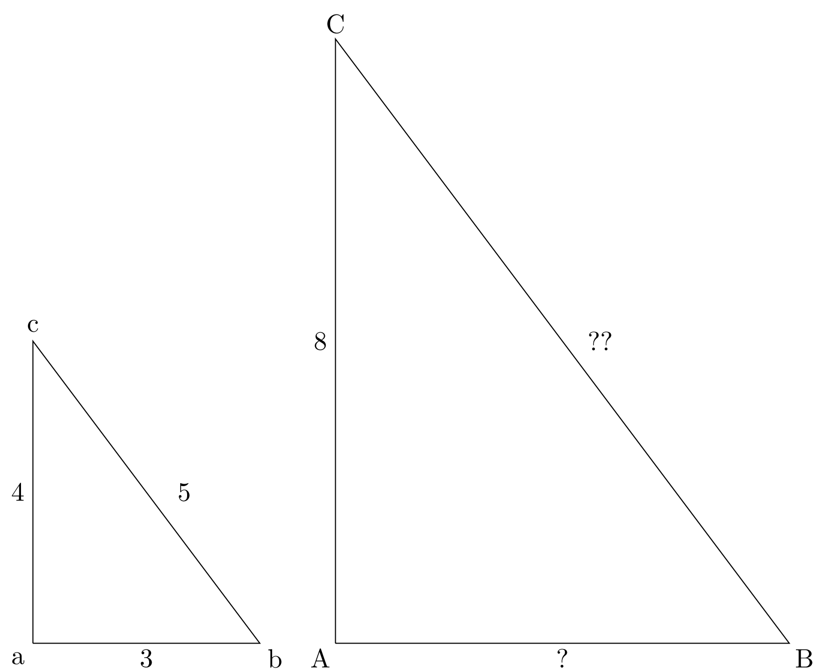


Figure 4.12: These two triangles are similar

Knowing that these two triangles are similar and that corresponding ratios of side lengths are the same in similar figures we can solve for the missing sides. This is what I mean:

First to solve for AB can consider the ratio $\frac{AB}{AC}$ the corresponding ratio in the small triangle is $\frac{ab}{ac}$ so we can set up our equation

$$\begin{aligned}\frac{AB}{AC} &= \frac{ab}{ac} \\ \frac{AB}{8} &= \frac{3}{4} \\ AB &= 6\end{aligned} \tag{4.5}$$

In the same way we can solve for the side length BC

$$\begin{aligned}\frac{BC}{AC} &= \frac{bc}{ac} \\ \frac{BC}{8} &= \frac{5}{4} \\ BC &= 10\end{aligned} \tag{4.6}$$

This is a powerful skill which has some fun applications. Shadows cast from different vertical object form similar triangles so you can use this to estimate height. More interestingly, photos of objects are necessarily similar to the objects themselves so you can use properties of similar objects to determine people heights from their pictures.

4.3 Pythagorean theorem

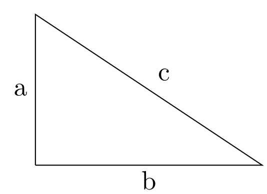


Figure 4.13: Right triangle

One lovely piece of Geometry that you may have heard of before is the Pythagorean theorem, named after Greek mathematician and Philosopher, Pythagoras. The statement of the theorem is very simple it says that, for a **right triangle** (a triangle with a right angle), the squares of the side lengths of the legs summed together is the square of the side length of the **hypotenuse** (The side opposite the right angle). In other words if you have a right triangle (Fig 13) then

$$a^2 + b^2 = c^2 \tag{4.7}$$

We can use this equation to answer questions about right triangles. For instance, If I have a ladder which is 18 ft long and I need to climb to a window 14 ft off the ground, how far from the base of the wall must I put the base of the ladder? We can imagine the wall and ground make a right angle so with the ladder we have a right triangle with the ladder as the hypotenuse. We have two of the three pieces of information in this equation so we can solve for the remaining piece of information as

$$14^2 + b^2 = 18^2 b^2 = 18^2 - 14^2 b = \sqrt{128} = 11.3 \tag{4.8}$$

This tells us that we need to put the ladder 11.3 ft away from the base of the wall. If you've ever read the safety material on a ladder you know that this is too far away from the wall and we really ought to use a different ladder to complete this job.

4.3.1 Proof of the Pythagorean theorem

There are very many visual proofs of this theorems, we will go through one in class. There are many others which are quick and available on youtube. You will not be required to recapitulate the proof.

Chapter 5

Topology

Topology is the study of surfaces and its study is a little bit different that the topics we've covered in previous chapters. You may be asking, how is this any different than geometry? Geometry is concerned with length and measure. Topology pays no mind to length and measure, rather we concern ourselves only with shape. When we do not concern ourselves with length we can say two solids are **topologically equivalent** even if the measures of length are not the same.

5.1 Genus

We can think about topologically equivalence by imagining clay which cannot rip or cut. In this way you can imagine any solid which has no holes in it can be squished to look like a sphere. Therefore we say any such solid is topologically equivalent. We call all the solids which are topologically equivalent to the sphere an **equivalence class**. The word equivalence class has huge mathematical implications. An equivalence class is a set made up of all the things which are equivalent to one another. When we talk about topological equivalence, an equivalence class is made up of all the different surfaces which can be made by morphing this imaginary clay without ripping or cutting it.

5.1.1 Example 1

Think about, now, a doughnut. You can stretch and squeeze this doughnut into lots of shapes like a coffee mug or a straw. All of these objects are characterized by having one hole. They form an equivalence class

What we see is that every surface with no holes is in one equivalence class, let's call it the sphere's equivalence class. Also all the surfaces with 1 hole are in an equivalence class which we could call the torus equivalence class. You can imagine that all of the solids with two holes are also topologically equivalent, as are all the solids with three wholes and so on. This way of numbering equivalence classes is called **genus**. The genus mathematically is the maximum number of non-intersecting closed paths I can draw on the surface so that the surface is not separated. We can show that this idea is the same as just counting the number of holes in a surface.

We use this definition: a **Jordan curve** is a simple continuous curve which does not intersect itself. We can restate our definition of genus now as the maximum number of Jordan curves that can be drawn on a surface so that the surface is not divided into two. This is a confusing definition but think of it like this: It is the maximum number of times you can cut the object clean through so that it is still in one piece.

Any way we cut a sphere clean through, it will be in two pieces so the genus of the sphere (and thus everything in its equivalence class) is 0. You can cut a doughnut clean through one time by having the tip of the knife pass through the center of the doughnut and it will still be in one piece, however if you try to cut it again it will result in 2 pieces so a doughnut is a solid of genus 1.

It turns out that the genus of a solid is equal to the number of holes that go *through* a solid. In a pair of non-ripped shorts, there are two holes that go through them so they are of genus 2. a sheet of paper that has been three hole punched has 3 holes so it is of genus three.

We cannot go much further in our discussion of genus without some pretty involved analysis so for now we can treat this as a beautiful piece of math trivia you can enjoy whenever you see a coffee cup or a doughnut.

5.2 The Möbius strip

The discussion of topology thus far has been about a type of shape we describe as "**orientable**." A surface is orientable if there is a consistent definition of left and right. This property is hard to understand because it is so inherent to the world we live in. Because all the surfaces we can touch are orientable, we can always say we know which way is right and which way is left. Another way to describe a surface as orientable is to say that if I walk in a closed loop along the surface and return to where I started facing the same direction, my right and my left will still be in the same direction. This is the standard way we walk around in the the world.

However, there are **non-orientable** surfaces, These would be shapes where, if an object moves in a closed path along the surface of the shape and returns to its starting point facing the same direction, left and right may be facing in different directions. We cannot make or draw a non-orientable shape outright, they really only exist theoretically. We can, however, make an approximation of such a shape. You may have heard a famous one: **The Möbius strip** (Fig 5.1).

The Möbius strip is a theoretical non-orientable two dimensional surface. We approximate it by taking a strip of paper, giving it a half twist, and taping the ends together. This is of course an approximation because we are using a three dimensional object in three dimensional space but it gives us some interesting insight into what it means to be non-orientable.

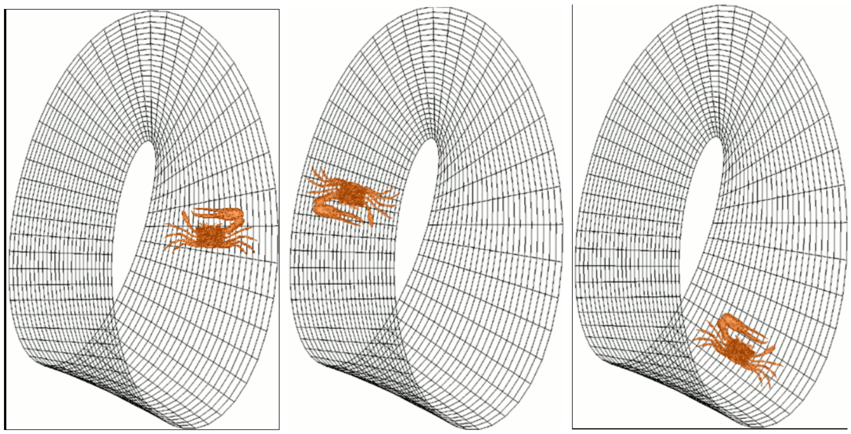


Figure 5.1: The non-orientability of the Möbius strip. Notice that the the large claw of the crab starts (on the left most image) closer to the outside edge. As the crab moves forward the large claw remains on the "outside edge" but as it flips over in the final image "outward edge" becomes the "inward edge" and the Large claw of the crab goes from being its right claw to its left claw. This is what it means to have an inconsistent notion of right and left.

One of the first things you'll notice about the Möbius strip is that, if you trace your pencil along it, it seems only to have one side even though we started with a two sided piece of paper. Like wise it only has one edge. We will do an experiment in class about what happens which you cut a Möbius strip down its "center." Before we do that make a hypothesis about what will happen to the two separate parts.

This ends our discussion of topology. Topology is a very beautiful branch of mathematics which can be appreciated all over. Although direct applications are limited to us, we can see its beauty all around us.

Chapter 6

Graph Theory

6.1 Introduction to graphs

In this Section we introduce a mathematical object which has shape, the **Graph**. A graph is a set of vertices and a set of edges which connect one vertex to another. A graph is in some sense the most simple mathematical object which

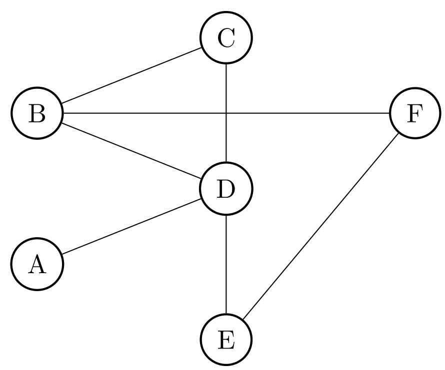


Figure 6.1: An example of a graph

has a concept of "shape." It's simplicity makes it a great object for discovery and investigation.

6.1.1 Example 1

In Figure 6.1 we see we have a set of **vertices** $V = \{A, B, C, D, E, F\}$ and a set of **edges** $E = \{AB, AD, BC, BD, BF, CD, DE, EF\}$. The Edge AB is simply the edge between vertex A and B.

For now we will deal with only **simple graphs** which are graphs whose edges have no direction or weight, which contain no **loops**, and with at most

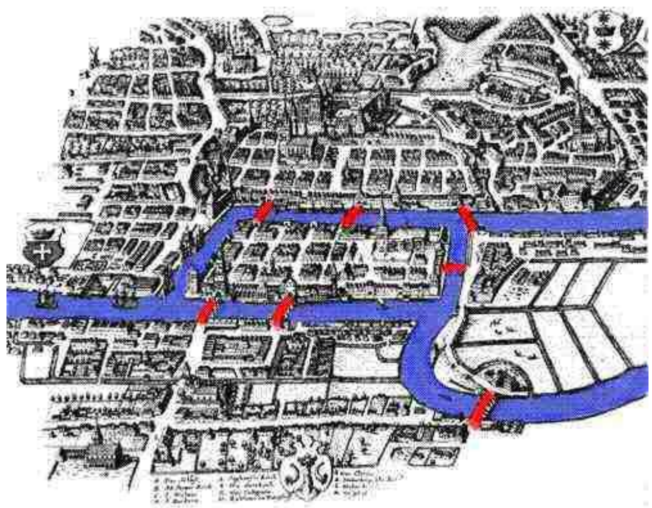


Figure 6.2: A map of Königsberg with bridges highlighted in red

one edge between two vertices. Loops here mean an edge from a vertex to itself. Observe above that the graph in Figure 6.1 satisfies these conditions. There are many other qualities that we will discuss about graphs but let us first discuss the origins of graph theory.

6.2 Königsberg bridge problem

Graph theory was invented by the mathematician: Euler, whose name you may have heard of before. In the list of the most famous mathematicians he is very close to the top. The story goes that Euler was walking around the city of Königsberg, which has many bridges. Seeking to entertain himself he asked if there was any path throughout the city which crossed every bridge exactly once.

In Figure 6.2 is a map of Königsberg with the bridges highlighted in red. You'll notice right away that this is not a simple graph because there are more than one bridge between parts of the city. Nevertheless this problem inspired the field of graph theory. We can turn this map into a graph (Fig 6.3).

By reducing the map to a graph we can see more clearly what possible paths may be taken. Although we haven't built up the way to solve this problem quite yet we can look at it and try to intuit a solution to Euler's brain teaser. Is there a walk that we can take which cross every bridge exactly once? The key to solving this problem is to notice that every vertex has an odd number of **neighbors**. Here, "neighbors" means the vertices with which a vertex shares an edge. Another helpful term is **degree**; the degree of a vertex is simply the number of neighbors that vertex has. Because every vertex has an odd number of neighbors (or every vertex has odd degree), in order to cross every bridge,

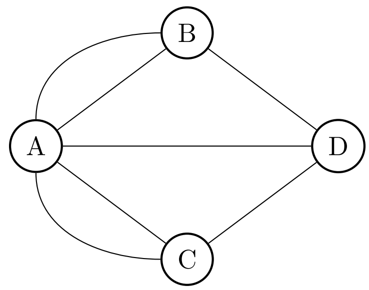


Figure 6.3: The city of Königsberg represented as a graph

wherever you start, you must leave come back and leave again. Therefore, you cannot end where you began. Now, think about a vertex at which you do not start. If you visit that vertex along your walk, you must leave it and return to it, because there is an odd number of neighbors. Furthermore, upon your return to that vertex you cannot leave again. This is true for any vertex so we cannot build such a path.

This solution is a bit of a preview for the kinds of problems and solutions we will consider in this chapter. We first need to consider some more tools.

6.3 Paths and circuits

We've been thinking about a path along a graph without having formally defined it. There are actually three types of "paths" which are each subtly different.

A **walk** is any sequence of edges which join a sequence of neighboring vertices. A **Trail** is a sequence of *distinct* edges which join a sequence of neighboring vertices. A **Path** is a sequence of edges which join a sequence of *distinct* neighboring vertices. In other words. a Trail is a walk where no edges repeat, and a path is a walk where no vertices repeat. On a simple graph every path is a trial because if a walk does not visit the same vertex twice it can never hope to use the same edge twice.

Having defined what a path is, lets us determine is a graph is **connected**. A graph is connected if there is a path from every node to every other node.

With this definition we can also ask some helpful questions. For instance, it may be important to know what is the shortest path between two vertices. The length of the shortest path between two vertices is what we call the **distance** between those two vertices. For instance in figure 6.4 the distance between A and F is 3. In symbols we write that as $dist(A, F) = 3$.

Once we know what the distance between vertices is we can give the definition of **diameter**, which is the maximum distance between any two vertices in a connected graph. In order to calculate this we need to find the distance between every pair of vertices. Once we have found all of those distances, the maximum distance is the diameter. For instance, in figure 6.4 the maximum distance between any two vertices is 3 so we say this graph has diameter 3.

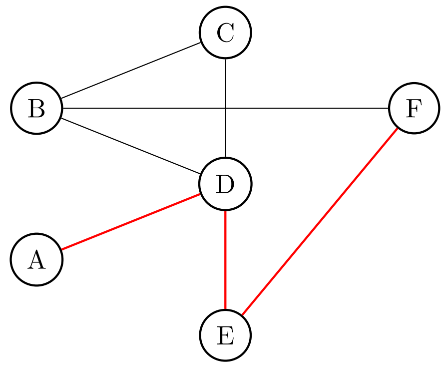


Figure 6.4: The Path from A to F is of length 3.

One question that may come to mind is: If there is a walk between vertices A and B is there necessarily a path between these vertices? The answer is yes. The proof of this is rather illustrative so we will go through it here.

6.3.1 Theorem

If there is a walk between vertices A and B in a graph then there is a path between A and B

6.3.2 Proof

Consider the walk between A and B. If this walk is already a path then there is a path between A and B. If this walk is not a path then it must repeat a vertex by the definition of a path. If it repeats a vertex, call the repeated vertex C and write it out as follows

$$\{Ax_1, ..., x_2C, Cx_3, ..., x_4C, Cx_5, ..., x_6B\} \tag{6.1}$$

we write xs here to indicate other unknown vertices. We don't know what these vertices are but we don't need to in order to complete the proof. Now that we have our walk written like this. Notice that if we were to clip out the section of the walk which starts and ends at C we would still have a walk

$$\begin{aligned} &\{Ax_1, ..., x_2C, \underbrace{Cx_3, ..., x_4C}_{\text{remove this section}}, Cx_5, ..., x_6B\} \\ &\{Ax_1, ..., x_2C, Cx_5, ..., x_6B\} \end{aligned} \tag{6.2}$$

We notice that this is still a valid walk but now it only visits the vertex C once. With this tool for removing a repeated vertex from a walk can repeat this procedure until there are no repeated vertices in the walk. A walk with no repeated vertices is a path so there is a path between A and B. Thus we know that in any case, if there is a walk between A and B then there is a path between A and B.

This proof illustrates for us how to take a walk and remove repeated vertices to make it a path. Removing repeated vertices may be helpful in graph theoretical problems because it leaves us with a walk that we might consider "more efficient". It also tells us how we can lengthen a walk by adding in a repeated vertex. This is also an interesting problem solving technique which can be helpful in games involving dominoes.

6.3.3 Remark

In the previous proof we cut out a piece of a walk which stated and ended at the same vertex. This deserves a definition. A **closed walk** is a walk that starts and ends at the same vertex. More usefully, a **circuit** is a trail which starts and end at the same vertex. Not every graph has circuits. We can ask what is the smallest or largest circuit in the graph and this tell us something about how easy or hard it is to move around a graph. For instance in Figure 18 the largest circuit is $\{BC, CD, DE, EF, FB\}$ and the smallest circuit is $\{BC, CD, DB\}$. It will always be true that the shortest a circuit in a simple graph can be is length 3. Spend some time convincing yourself that this is true by thinking about what a circuit of length two would look like.

There are some special kinds of trails and circuits that we may like to search for. An **Eulerian** trail or circuit is a trail or circuit which uses every edge exactly once. This is named after Euler's quest to cross every bridge in Königsberg exactly once. Lets examine the graph we've been using and determine if there are any Eulerian circuits or trails. (Fig 6.5)

6.3.4 Example 1

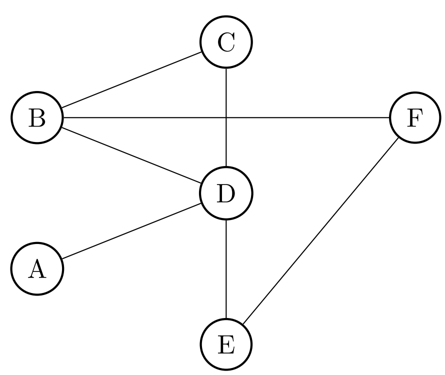


Figure 6.5: Does this graph contain any Eulerian trails or circuits?

We have already decided that the largest circuit in this graph is of length 5, there are more than 5 edges in this graph so there is no Eulerian circuit. Moreover we can say that if any graph has a vertex with an odd number of neighbors it cannot have an Eulerian circuit. Why do you think this is?

The "proof" requires us to imagine making a eulerian circuit on a graph with a vertex with an odd number of neighbors. Suppose the vertex A has an odd number of neighbors. Imagine the circuit arrives at A for the first time, now there are an even number of edges left to use in the circuit. However, to use an even number of edges we need to leave from A and return to A the same number of times. This means that once we arrive at A, if we want to use every edge, we will eventually get "stuck" at A, meaning we will arrive at A and we will not have any more edges to use to get away from A. Because of this fact, If a graph has a vertex with odd degree, that graph cannot have an Eulerian circuit.

6.3.5 Example 2

Now lets examine Fig 6.5 for any Eulerian trails. We can see that for a trail to include every edge exactly once it must start or end at A. If not, as soon as our trail takes the edge DA we are stuck and cannot return to the rest of the graph. So lets start at the vertex A on edge AD and build our trail from there. Lets next take edge DE then we have to take the edge EF. From there we still have no choice but to take the edge FB. Once we are at vertex B we again have to decide what to do next. Lets take BD next. The only edge remaining from D is DC so we take that and finally we must take BC. Now we can check that this trail has indeed crossed every edge exactly once.

$$\{AD, DE, EF, FB, BD, DC, BC\} \tag{6.3}$$

Once we confirm that it has, we can say that this is an Eulerian trail. It is not the only Eulerian trail, there are many others in this same graph. Also we should note that not every graph will have an Eulerian trail.

To conclude this section we state two important results about Eulerian trails and circuits.

6.3.6 Theorem 1

A connected graph has a Eulerian circuit *if and only if* it has no vertices of odd degree.

6.3.7 Theorem 2

A connected graph has a Eulerian trail *if and only if* it has exactly zero or two vertices of odd degree. Moreover the Eulerian trail must start and end at the vertices of odd degree if they exist.

This second theorem gives us an idea of how to find an Eulerian trail and both theorems give us a powerful tool to determine if an Eulerian trail or circuit is possible. In general finding Eulerian trails and circuits can be tricky and, interestingly, questions like these can be hard for computers to do very quickly.

6.3.8 Remark

There is a way to find an Eulerian circuit which is not too difficult for small graphs. you start by identifying any circuit in a graph. If there are edges left in the graph then, if the graph has vertices of all even degree, the remaining edges must can be made into circuits as well. If they are spliced together carefully then we will get an Eulerian circuit. The same is true for the Eulerian trail as long as our trail starts and ends at vertices of odd degree. The details of this algorithm are will be discussed in class if time allows but, for the graphs we will encounter in this class, it ends up not being much faster than just searching for a Eulerian trail or circuit outright.

6.4 Complete graphs and spanning trees

There are some graphs which are so important they are given special names. They are important because they come up in nature or computing very frequently. They also serve as extreme examples of what graphs can be and so they are often useful in proving things about graphs.

6.4.1 Complete Graphs

The first of these important graphs we will talk about are the **complete graphs**. The complete graph is the graph where every vertex is connected to all of the other vertices. The complete graph on n vertices is named K_n. Below are a couple examples of complete graphs (Fig 6.6). This is somehow the "most

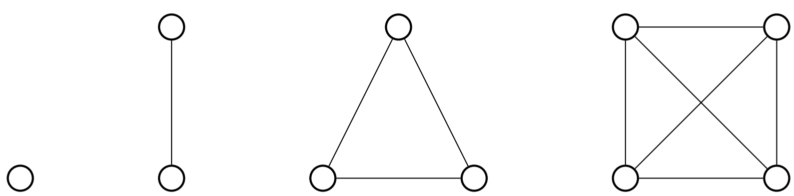


Figure 6.6: Pictured from left to right are K_1, K_2, K_3, and K_4

connected" graph. The diameter of any complete graph is 1 because the shortest path from any one vertex to any other vertex is 1. It is interesting to ask whether there are Eulerian trials and Eulerian circuits in these graphs. Using the two theorems which concluded the previous section we can say that K_2 has an Eulerian trail but no Eulerian circuit because it has two vertices of odd degree. K_3 has an Eulerian trail which is an Eulerian circuit because it has zero vertices of odd degree. Furthermore, K_4 cannot have and Euler trail because it has 4 vertices of odd degree. In general we can say that K_n has an Eulerian circuit if n is odd but not if n is even.

6.4.2 Trees

Whereas the complete graph was the "most connected" connected graph, lets consider now the "least connected" connected graph. As an exercise try to imagine a graph on 6 vertices which is connected but would become disconnected if you removed any of the edges.

The solution to this problem is a type of graph that we call a tree. A **tree** is a connected graph wherein, if any of the edges were removed, it would become a disconnected graph. Equivalently and more clearly it is a connected graph that has no circuits. Trees are very important because they are somehow the "least connected" connected graphs. There are many kinds of trees but they all share the same feature, they do not contain any circuits. Below are pictured several trees (Fig 6.7).

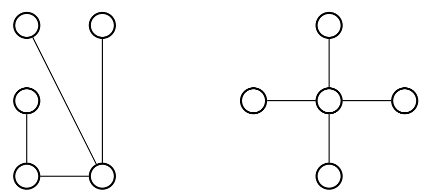


Figure 6.7: Both graph shown here are trees because they contain no circuits

We can think of trees as the "least connected" connected graphs. They are very useful because of this fact. It may be useful to remove redundancies in a graph. Just as when we removed redundancies in a walk to get to a more efficient path, we can remove redundancies in a graph to get a tree. The idea that we are approaching here is called a **Spanning tree**. A spanning tree of a graph G is a tree which has all the vertices of G and a subset of the edges of G. a **subset** means we pick some but not necessarily all of the original elements. Thus when we say a subset of the edges we mean some but not necessarily all of the original edges.

6.4.3 Example 1

So a spanning tree of a graph is a tree on the vertices of the original graph using some but not necessarily all of the edges of the original graph. Below is pictured an example of a spanning tree (Fig 6.8). We notice in the figure that there can be multiple spanning trees.

We can practice finding spanning trees using a simple algorithm. For any connected graph we can find a spanning simply by removing edges (carefully) one by one until we are left with a tree. We start by taking any edge, we ask "if I remove this edge will the graph still be connected?". If the answer is yes,

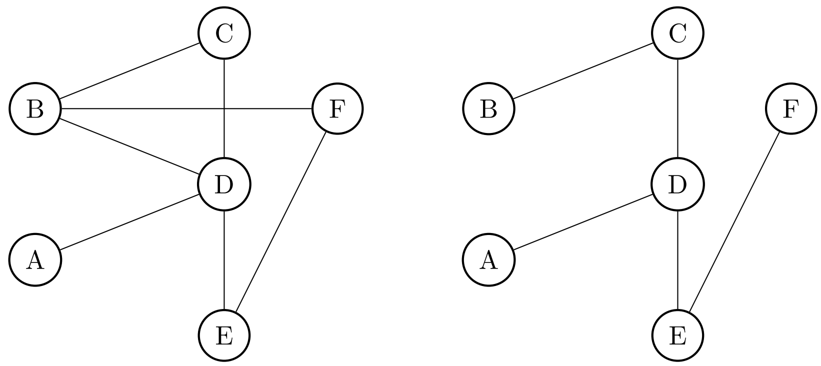


Figure 6.8: on the left is a graph and on the right is a spanning tree of that graph. Notice that the spanning tree has all of the vertices and some but not all of the edges of the original graph

then remove the edge and repeat the process on another edge. By repeating this process we will result in a connected graph on the original vertices using a subset of the original edges and, if we've done our process carefully, if we removed any of the remaining edges the graph would become disconnected. This means that the new graph is a tree (Recall our earlier exercise).

In Figure 6.9 we show an example of how to find a spanning tree. The algorithm above is equivalent to searching for any circuits in the graph, if you find a circuit, remove one of its edges so that it is no longer a circuit. If we repeat that procedure we will eventually get to a graph which has no circuits. As we know a graph with no circuits is a tree so we will have succeeded in finding a tree on the vertices of the original graph with a subset of the original edges.

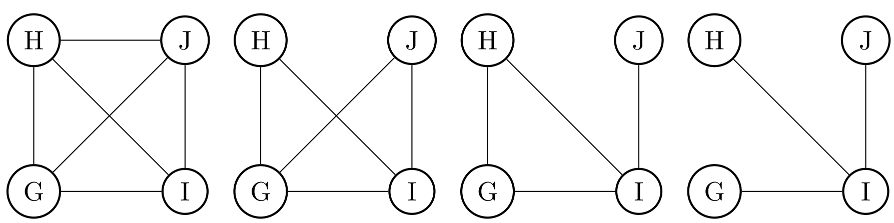


Figure 6.9: Proceeding from left to right we start with the graph K_4. We select the edge HJ and, because if we remove it the graph will still be connected, we remove it. Next we select the edge GJ and, for the same reason, we remove it. In the last step if we want to remove JI we cannot because it would leave the graph disconnected but we can remove the edge GH so we do that. On the right we are left with a spanning tree of our original graph.

6.5 Planar graphs and Euler's characteristic theorem

We have yet to talk about the geometry of a graph. So far our discussion of graphs have been impartial to the way that the graphs are drawn. However, as you have drawn graphs you may have found it tempting to try to draw them without having edges intersect. This is related to the main Idea of this section: planar graphs.

A graph is **planar** if and only if it can be drawn in two dimensions without any of the edges intersecting.

Note that this represents a shift in how we think about graphs. Previously we thought of graphs without geometry but now we must think of them as real objects on the a plane. Drawing the graph on a 2 dimensional surface like a whiteboard or piece of paper is an appropriate way of representing a graph in two dimensions. Such a drawing of a graph is called a **Planar embedding**.

While there is an easy way to determine if a graph is planar, it is beyond the scope of this course, instead we will focus on drawing planar embeddings. To draw a planar embedding we want to imagine moving the vertices around so that their edges do not intersect. Consider the following example with K_4 (Fig 6.10)

6.5.1 Example 1

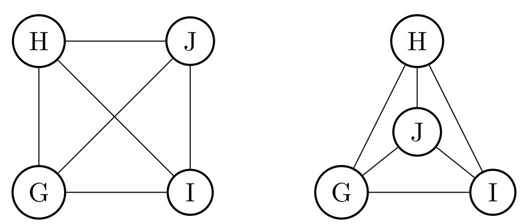


Figure 6.10: If we take K_4 and move the vertex J so that JG and HI no longer intersect, then we get a planar embedding

As far as I know, there is no general algorithm for finding planar embedding but our goal will always be to rearrange the vertices so that no vertices intersect.

6.5.2 Example 2

In the graph we have been using as an example throughout this section, lets try to find a planar embedding. (Fig 6.11) We will, in some cases, need to move many vertices around to find a planar embedding but if a graph is planar we will always be able to find a planar embedding.

There is another very interesting thing about planar graphs but in order to talk about it we need to give a definitions. A **Face** is a polygon in a planar

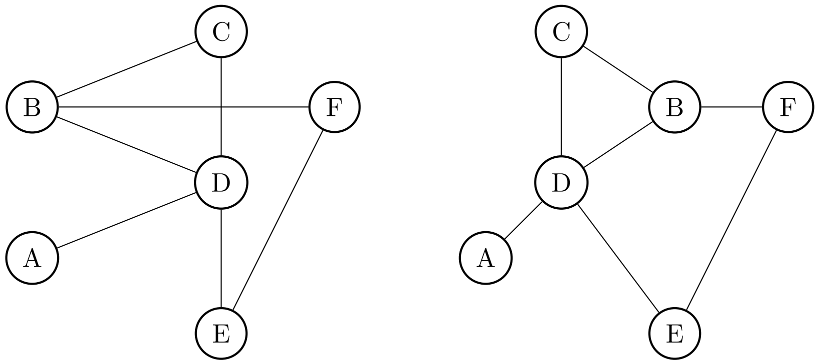


Figure 6.11: On the left is the original graph. We notice that the edges BF and CD are intersecting so we need to rearrange one of those vertices draw our planar embedding. if we move B closer to F we get a planar embedding, importantly this is not the only valid planar embedding.

embedding formed by edges. For instance, in figure 6.11 the polygon $BFED$ is a face, the polygon CBD is a face. We will also include the space outside of the graph as a face so the planar embedding in figure 6.11 has 3 faces. Likewise the planar embedding in figure 6.10 has 4 faces. Armed with this definition we can give one of the most interesting theorems in graph theory.

6.5.3 Theorem

Euler's Characteristic Theorem says that for any connected planar graph, if V is the number of vertices, E is the number of edges, and F is the number of faces in the planar embedding of the graph, then

$$V - E + F = 2 \tag{6.4}$$

6.5.4 Remark

It is quite remarkable that such a complicated object has such a constant and simple relationship. In the polar embedding in figure 6.11 and we will see that there are 6 vertices, 7 edges and 3 faces and indeed, $6-7+3=2$. This is always true for any planar graph.

Excitingly, we know from our discussion of topology that different surfaces have different behavior. A plane (which can really be thought of as a sphere) has genus 0. If we try to embed a graph on something which is not a plane, for instance a torus, we will get a different constant in our equation. This is, for our purposes, a piece of trivia which we cannot use well but we can say a graph drawn on a torus has $V - E + F = 0$. If it is drawn on a double torus (a torus with two holes) it has $V - E + F = -2$. If it is drawn on a Möbius strip it has $V - E + F = 0$.

6.6 Coloring and the Four Color Theorem

The last interesting thing we will discuss in graph theory is coloring. Think of a political map. The image that comes to mind is an image of countries which are colored different colors. Importantly, territories are colored so that they are a different color from all of their neighbors. This is a graph coloring

A **proper coloring** of a graph is when each vertex is designated by a color, and no neighboring vertices have the same color. Consider the below example(Fig 6.12)

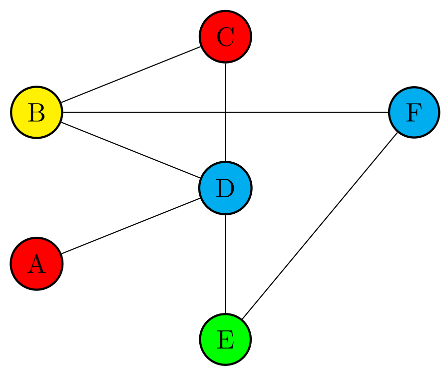


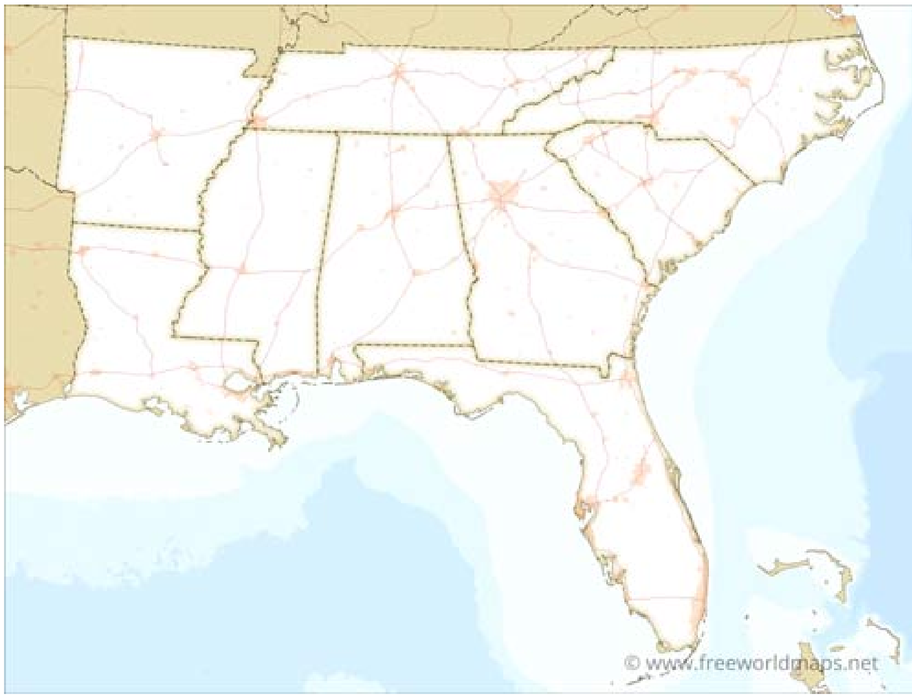
Figure 6.12: Notice in this proper coloring that no two neighboring vertices share the same color.

As I suggested in the introduction to this section, this is most easily seen when we consider drawing maps. Consider the map of the states in the south east (Fig 6.13). We can express this as a graph where every state is a vertex and two vertices share an edge if their corresponding states share a border. To give a proper coloring of this map we need only start by designating one vertex a particular color then moving to its neighbors.

One way to color it properly is to just give every vertex a different color. This certainly is a proper coloring. The more interesting question is find colorings with a limited number of colors. Let us try to color the map of the American south east with only four colors: Red, Green, Cyan, and Yellow.

6.6.1 Example 1

Lets start by designating FL as red, FL neighbors GA and AL so neither of these can be red. Furthermore AL and GA are neighbors of one another so we cannot give them the same color. If we give AL green and GA cyan we are still on our way to a proper coloring (Fig 6.14). Next we can see that TN can still be colored red. If we do that NC cannot be red or cyan but we can color it green and likewise SC cannot be green or cyan but we can color it red. To the west, we can choose to color AR cyan which means the only color remaining for MS



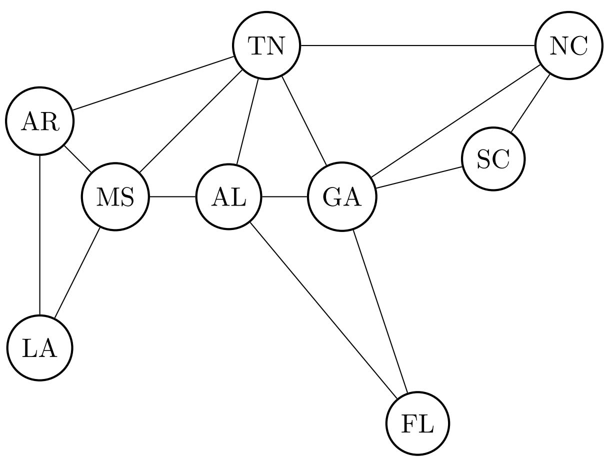


Figure 6.13: Above is an empty political map of the south eastern united states. If we consider each state a vertex and let two vertices share an edge if the states share a border then we can translate the map into this graph

is yellow. Lastly we need only color LA with either available color, I've chosen green.

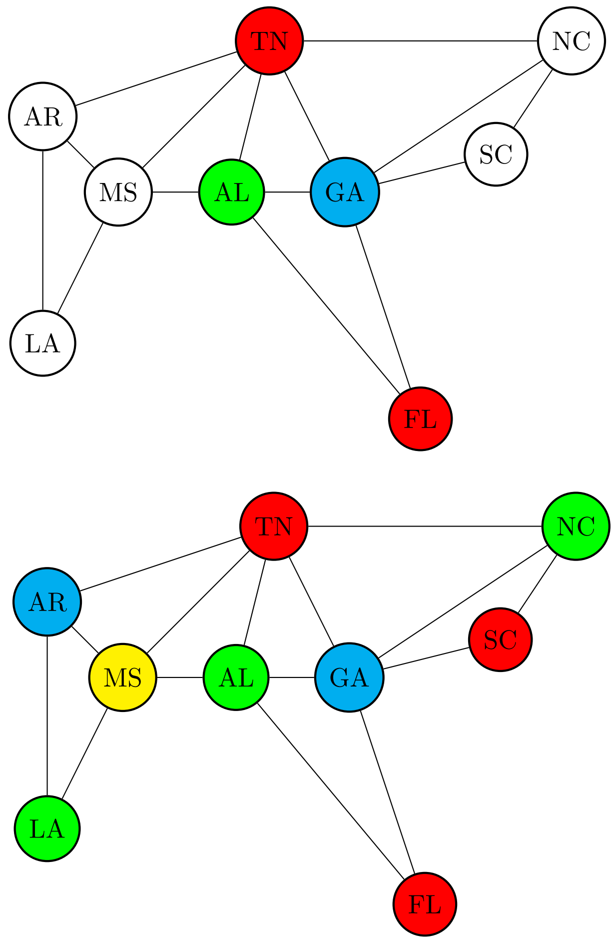


Figure 6.14: We started by coloring FL red then both of its neighbors get a different color. We continue to color the rest of the vertices so that no vertex has the same color as its neighbor.

6.6.2 Example 2

Finding colorings of graphs can be useful for more than just map making. It is also used in some schedule making algorithms, For instance if a group of teachers are deciding which classrooms to teach in, we can let vertices represent teachers and let teachers be connected by edges if they teach at the same time. If we assign a proper coloring to the graph where each color represents an available classroom then we can ensure that no two teachers will need the same classroom at the same time.

It is not a coincidence that both examples (Figs 6.12, 6.14) have been able to be colored with only 4 colors. Indeed there is a very powerful result in graph theory that tells us this is always possible. It is creatively named the four color theorem.

6.6.3 Theorem 1

four color theorem Every planar graph can be given a proper coloring with no more than 4 colors.

Proving this theorem would be beyond the scope of even a dedicated graph theory course, it requires that we check 1936 cases which we simply do not have time for. It is nevertheless a remarkable theorem; it means that for any planar graph you can always find a proper coloring of the vertices with only four colors. The theorem does not give any indication of how to find such a coloring. Indeed there is no known algorithm for finding minimal colorings (colorings with the fewest possible colors), all the theorem says is that there certainly is one.

Chapter 7

Probability

7.1 Introduction

In the following section we discuss probability. The idea of probability is something we are familiar with even if we have not learned it in a formal capacity. You know when something is unlikely and you know what it means for something to be probable. Our goal here is to be able to write down probabilities and make probabilistic computation.

In order to do this we must first lay the ground work with some definitions: Suppose there is an **experiment** whose result we do not know for sure. We use probabilities to describe it. Each possible result of the experiment is called an **outcome**. We say an **event** is a subset of the outcomes. Each outcome that leads to that event is sometimes called a **favorable outcome**. The **probability** of an event is a measure of how likely an event is and it is given like this:

$$P(\text{Event}) = \frac{\text{Number of favorable outcomes}}{\text{Number of total outcomes}} \tag{7.1}$$

7.1.1 Example 1

Suppose that I have two coins and I flip both of them. The possible outcomes are: Both come up heads (H, H), the first is heads and the second is tails (H, T), the first is tails and the second is heads (T, H) or they may both be tails (T, T). These are all the possible outcomes Notice that the outcomes (H, T) and (T, H) might be indistinguishable to us if both coins are flipped at the same time but they are still separate outcomes.

7.1.2 Example 2

How likely is it that the coins do not match? In this case the event that I am interested in is $\{(H, T), (T, H)\}$ because either of these outcomes produce the result in question. Thus the probability of this event is given as

$$P(\text{Coins don't match}) = \frac{2}{4} = 0.5. \tag{7.2}$$

7.1.3 Example 3

I could also ask how likely is it that both coins show heads. In this case the event is question is $\{(H, H)\}$. No other outcome is favorable. Thus the probability is given by

$$P(\text{Both heads}) = \frac{1}{4} = 0.25 \tag{7.3}$$

7.1.4 Example 4

Lets move to a different experiment. For this example we need to discuss a deck of cards. To make sure we are all on the same page, lets first recall that a deck of cards has 52 cards divided into four suits: Hearts$\heartsuit$, Diamonds$\diamondsuit$, Clubs$\clubsuit$, and Spades$\spadesuit$. Each suit has 13 card which are labeled 2,3,4,...,10,J,Q,K,A. To talk about the two of spades I will say 2$\spadesuit$. Likewise, to talk about the Jack of hearts I will write $J\heartsuit$.

Now suppose two people each have a deck of cards that is well shuffled (In our class "we will assume that every deck of cards is well shuffled). Both people draw the top card from the deck. Before we ask a question let us think about all the possible outcomes. Player 1 may draw any of 52 cards in their deck and Player 2 may also draw any of the 52 cards in his deck so if we start listing outcomes it might look something like this

$$\begin{Bmatrix} (2\clubsuit, 2\clubsuit) & (2\clubsuit, 3\clubsuit) & ... & (2\clubsuit, A\spadesuit) \\ (3\clubsuit, 2\clubsuit) & (3\clubsuit, 3\clubsuit) & ... & (3\clubsuit, A\spadesuit) \\ \vdots & \vdots & \ddots & \vdots \\ (A\spadesuit, 2\clubsuit) & (A\spadesuit, 3\clubsuit) & ... & (A\spadesuit, A\spadesuit) \end{Bmatrix} \tag{7.4}$$

It would be very tedious to count out each of these outcomes but with some though we can conclude that there are $52 \cdot 52$ total outcomes because both players have 52 cards to choose from. From this list of total outcomes can we determine the probability of both players picking the same card?

In this case, the event in question would look like this

$$\{(2\clubsuit, 2\clubsuit), (3\clubsuit, 3\clubsuit), ..., (A\clubsuit, A\clubsuit), (2\diamondsuit, 2\diamondsuit), ..., (A\spadesuit, A\spadesuit)\} \tag{7.5}$$

Listing them out like this we can see that there are 52 favorable outcomes, so we can calculate the probability as

$$P(\text{Same Card}) = \frac{52}{52 \cdot 52} = \frac{1}{52} \tag{7.6}$$

This example is tedious to find because we are dealing with such a huge set of outcomes. We can find this probability easily enough, but If I asked "How likely is it that Player 1 draws a heart and Player 2 draws a spade" then it becomes a little more difficult to calculate directly in this way. For this we take advantage of something called "independent probabilities."

7.2 Independent probabilities

Events are said to be **independent** if one has no effect on the other. (This is more of an intuitive definition than a mathematical definition. A more thorough probability class will discuss a mathematical definition of independence) Take the card experiment for example. The card that Player 1 draws has no effect on the card that Player 2 draws so we can consider these two actions independently. This is helpful because we can use the following rule.

7.2.1 Theorem

Law of independent probabilities: Suppose that A and B are independent events then

$$P(A \text{ and } B) = P(A) \cdot P(B) \tag{7.7}$$

7.2.2 Example 1

If I ask what it the probability that player 1 draws a heart and player 2 draws a spade, I can interpret this as two independent events. The first is that player 1 draws a heart and the second is that Player 2 draws a spade.

$$P(\text{Player } 1\heartsuit \text{ and Player } 2\spadesuit) = P(\text{Player } 1\heartsuit) \cdot P(\text{Player } 2\spadesuit) \tag{7.8}$$

Player 1 will draw a heart with a probability 13/52 because there are 13 hearts in the deck of 52 outcomes. Likewise Player 2 will draw a spade with a probability 13/52 because there are 13 spades in the deck of 52 outcomes.

Thus we see that $P(\text{Player 1 draws a heart and player 2 draws a spade}) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$.

7.2.3 Example 2

The rule of independent probabilities can be very helpful in simplifying problems with lots out outcomes. For instance, if a bag has 100 marbles in it and 5 of them are blue, the rest are green and you are asked to pick a marble out of the bag, put it back, and pick another one, we may think of each time you pick a marble as an independent action. So, if I ask you how likely is it that you pick a blue marble twice in a row, we can think of this as

$$P(\text{Two blue marbles in a row}) = P(\text{Picking a blue marble}) \cdot P(\text{picking a blue marble}) \tag{7.9}$$

The probability of picking a blue marble is 5/100 because there are 5 blue marbles out of a total of 100 marbles. Thus we have that the probability of picking two blue marbles in a row (With replacement) is $\frac{1}{20} \cdot \frac{1}{20} = \frac{1}{400}$

Now notice that if you did not put the first marble back after you took the next marble then these probabilities would not be independent. The marble you take out first effects the marble you take out next. We can still analyze this problem but we cannot use the rule of independent probabilities like we did before.

7.3 Expected value

Sometimes outcomes of an experiment are associated with a value and we may want to know what value we can expect from an experiment even if we do not know which event will occur. This is the idea of **expected value**. It can be thought of as the average value of the experiment.

7.3.1 Example 1

The canonical example of expected value is the lottery. Suppose you enter a lottery and stand to win \$300,000. However, there are one million people total playing the lottery. If each person stands an equal chance at winning, you will win with probability $1/1,000,000$. You will loose with probability $999,999/1,000,000$. Your expected payout is calculated as the amount you stand to win times the probability that you win, plus the amount you get if you don't win times the probability that you don't win. In this case that would be

$$E = 300,000 \cdot \frac{1}{1,000,000} + 0 \cdot \frac{999,999}{1,000,000} = 0.3 \tag{7.10}$$

You are only expected to win \$0.30. If the ticket itself costs one dollar you are expected to loose money on your purchase.

More generally we can say that the expected value of an experiment is the sum of the possible values multiplied by their likelihoods. Another way to say that is that the expected value is the average of all the possible values weighted by their likelihoods. If an experiment has possible values $v_1, v_2, ..., v_n$, the expected value of the experiment is $E = v_1 P(v_1) + v_2 P(v_2) + ... + v_n P(v_n)$. where $P(v_i)$ is the probability of achieving that value.

7.3.2 Example 2

If you role a fair 4 sided die (d4) what is the expected value? Each number from 1 to 4 will come up with probability 1/4 so we can write

$$E = 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{4} + 4 \cdot \frac{1}{4} = \frac{1}{4}(1+2+3+4) = 2.5 \tag{7.11}$$

7.3.3 Example 3

What if we roll 2 d4s and sum them up? Now the calculation is not as easy but it is still doable. We need to first consider all the outcomes and determine their value. Once we calculate all the possible values then we can find their associated probability.

Outcome	Value
1,1	2
1,2	3
1,3	4
1,4	5
2,1	3
2,2	4
2,3	5
2,4	6
3,1	4
3,2	5
3,3	6
3,4	7
4,1	5
4,2	6
4,3	7
4,4	8

Value	Probability
2	1/16
3	2/16
4	3/16
5	4/16
6	3/16
7	2/16
8	1/16

With all This we can calculate the Expected value by adding up all of the possible values multiplied by their probabilities.

$$E = 2 \cdot \frac{1}{16} + 3 \cdot \frac{2}{16} + ... + 8 \cdot \frac{1}{16} = 5 \quad (7.12)$$

We have thus calculated that the expected value of this experiment is 5.

7.4 Permutations and combinations

More complicated probabilities require some more sophisticated tools. Permutations and combinations are those tools.

7.4.1 Permutations

A **permutation** is a reordering of a set for instance $\{2, 4, 3, 1, 5\}$ is a permutation of the natural numbers between 1 and 5. It is often helpful to know how many permutations there are for a given set.

If we seek to reorder the set $\{1, 2, 3, 4, 5\}$ we first ask which number goes in the first spot. We could choose any of the 5 numbers available so there are 5 possibilities. After we've placed a number in the first spot there are 4 remaining numbers to be placed in the second spot. After that there are 3 remaining numbers to be placed in the 3rd spot, 2 remaining which could be placed in the second spot and by the time we reach the last spot there is only one number remaining so it must be placed there. If we think about all of the choices we had, we can find the total number of possible permutations. We first chose from a set of 5 possibilities, then from a set of 4 possibilities, then from 3 then from 2 then from 1. Therefore the total number of permutations we could have made is given by $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$. We have a special symbol for this, called the **Factorial**. $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$. In general there are $n!$ permutations of a set of n different objects.

7.4.2 Example 1

If you and your 5 friends are entering a bar and you line yourselves up randomly to have your ID checked, how many ways might you line yourselves up? We are asking about an ordering of 6 people so there are 6! possible permutations.

7.4.3 Example 2

Now lets use this tool to calculate a probability. During the pandemic "marble racing" became very popular in some corners of the internet. Assume marble racing it totally random and independent of the order the marbles start in. If there are 7 marbles: Red, Orange, Yellow, Green, Blue, Indigo, and violet. How likely is it that they finish exactly rainbow order? There are 7! Possible permutations so there are 7! possible outcomes and only 1 favorable outcome in our case so the likelihood is $\frac{1}{7!}$.

7.4.4 combinations

Now that we understand permutations we can ask a more complicated question: If I have a group of n objects, how many different *unordered* groups of k can I make from those n objects? As an example if I have a group of 8 objects how many groups of 3 can I choose? We start in the same way as before. If we pick a first element in our group of three we have 8 things to choose from. Then for the next spot we have 7 things to choose from. Finally for the last spot we have 6 things to choose from. Therefore there are $8 \cdot 7 \cdot 6$ ways to choose and *ordered* group of 3 from a set of 8 things. Notice that I can write this as $\frac{8!}{5!}$. This is the number of ordered groups we can pick but if I don't care about the order, I need to adjust my calculation somehow. How many ways are there to order my group of 3? The answer we know is 3!. This step is complicated but we can divide the number of ordered groups of 3 by the number of ways to order that group of 3 to get to the number of unordered groups of three: $\frac{8!}{3!5!}$.

This example derivation may be hard to follow but the punchline is this: If you have a set of n things and you are trying to pick a group of k objects from that set, we count the number of possible ways to choose these groups as "n choose k" (${}_nC_k$). We have a symbol for this $\binom{n}{k}$ and we calculate it as

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{7.13}$$

7.4.5 Example 3

We can use this to solve some more complicated probabilities. If a teacher is splitting a class into groups of 4. How likely is it that the teacher chooses you and your 3 friends to be in the same group? If the class has 20 people in the class then we know there are $\frac{20!}{4!16!}$ ways to choose a group of 4. There is only 1 favorable group so the probability of picking this favorable group is

$$P(\text{Group with your friends}) = \frac{1}{\frac{20!}{4!(16)!}} \tag{7.14}$$

Chapter 8

Financial Math

8.1 Markdown

We start our discussion of financial math with a discussion of markdown. If you buy an item on sale, **Markdown** is a measure of how much the price is reduced. If you say that an item is 20% off then we say that the mark down is 0.2. This idea is not a difficult concept but because we can ask questions in so many different ways. The standard way to write this is

$$N = P(1 - d) \tag{8.1}$$

where N is the new price, P is the original price, and d is the mark down. Just as we are used to from algebra, we can be given two pieces of information and we can fill in the the rest of the information.

8.1.1 Example 1

For instance, if an object was originally \$15 and is marked down 5% then we can find the new price

$$N = \$15(1 - 0.05) = \$15 \cdot 0.95 = \$14.25 \tag{8.2}$$

8.1.2 Example 2

Another way to ask is if an item which is being marked down 40% now costs \$80, what was the original price? Again we use the equation.

$$\$80 = P(1 - 0.4) = 0.6 \cdot P \Rightarrow P = \frac{\$80}{0.6} = \$133.34 \tag{8.3}$$

This computation is can feel straight forward but depending on what information we have and what information we want, it can be hard to decipher how to proceed with the equation.

8.1.3 Example 3

For instance, If I ask about how much money you save when you buy and item which would have originally cost \$50 at a mark down of 15%, we know the original price and the mark down so we can use that to calculate the new price as before

$$N = \$50 \cdot (1 - 0.15) = \$42.5. \quad (8.4)$$

However we need to remember that the questions asked for how much was *saved* not the new price. The savings is equal to the original price minus the new price. Thus we have \$7.50 in savings.

8.1.4 Example 4

Sometimes sales aren't given in terms of markdown and so it can be helpful to calculate the mark down for a particular sales. For instance the boot barn in Pigeon Forge advertises a sale: Buy 1 pair of boots get three free. What is the mark down in this sale? We don't even need to know how much one pair of boots costs, we can just let x be the price of one pair of boots. Thus the original price for 4 pairs of boots is $4x$ and the sale price for these four boots is just x. Thus we can use our equation so find the mark down associated the sale.

$$x = 4x(1 - d) \quad \Rightarrow \quad \frac{1}{4} = 1 - d \quad \Rightarrow \quad d = 0.75. \quad (8.5)$$

So we could frame this sale as a 75% off sale.

8.2 simple interest

The world of finance can be very confusing but one of the crucial pieces of the finance world is the idea of interest. Money now is, in general, worth more than money in the future so from this we get the idea of interest. If I lend you money to be payed back after a year. Amount of money I loaned you was worth more at the time I lent it to you than it will be at the time the lone comes due. To account for that I may change interest so that the amount you pay back is worth the same as (or more than) the amount I loaned you was worth when I lent it to you. Likewise, if you make an investment, you are practically loaning that money to a company or an organization so you should be payed interest on that investment.

There are two ways we can calculate interest, the first is simple interest. If you borrow a an amount of money, that original amount that you borrowed is called the **principle**. In simple interest the lender adds some percentage of the principle to the loan at a regular time interval. That percentage is called the **interest rate** and the time interval is called the **period**.

If you have a loan with annual simple interest with an interest rate of 5% then 5% of the original principle is added to the loan at the end of each year. Suppose that the principle is \$1000. 5% of 1000 is 50 so after one year the borrower owes \$1050, at the end of the next year the borrower owes \$1100 and so on. The amount of the loan is increasing by the same amount each period so this should remind you of an arithmetic sequence. The initial condition is the principle and the common difference is the interest rate times the principle.

$$A_t = P + Prt = P(1 + rt) \tag{8.6}$$

where A_t is the value of the loan after t years, P is the principle and r is the interest rate.

In finance, we always measure time in years so A_t always means the value of the loan after t years. If the period of the lone is not a year, (e.g. a month or quarter), then interest is added more frequently. However interest rates are almost always given as an annual figure even if the period of the loan is not a year. So if the annual interest rate is 5% then the monthly interest rate is $\frac{5\%}{12}$ because there are 12 months in a year. These two particularities work to cancel each other out in this case. After t years there have been $t \cdot n$ periods if n is the number of periods in a year.

$$A_t = P\left(1 + \frac{R}{n} t \cdot n\right) = P(1 + Rt) \tag{8.7}$$

Where again A is the value of the loan after t periods, P is the principle, R is the annual interest rate, and n is the number of periods in a year.

8.2.1 Example 1

As an example suppose you borrow \$25,000 to buy a car and that loan has a simple annual interest rate of 4%. If you don't pay any of the loan back for 3 years how much will you owe at that time? We need only plug into our equation to see

$$A_3 = \$25000\,(1 + 0.04 \cdot 3) = \$28,000 \tag{8.8}$$

The value of the loan is \$28,000.

8.2.2 Example 2

As in the introduction of this section, making an investment can be frames as lending money so take as a second example an investment which has quarterly simple interest. Suppose that you invest \$5,000 and leave the investment alone for 8 years. How much will it be worth at the end of that time if the *annual* interest rate is 3%? First recall that the period of the loan is one quarter (one fourth of a year) in 8 years there are $8 \cdot 4 = 32$ quarters. Therefore we can write

$$A_8 = \$5,000\left(1 + \frac{0.03}{4} 8 \cdot 4\right) = \$6200 \tag{8.9}$$

8.3 Compound interest

The other kind of interest we will discuss here is compound interest. This is the far more common but slightly more complicated version of interest wherein the amount of interest accrued depends on the amount of interest previously accrued.

In compound interest, the value of the lone increases by the same proportion each period. Suppose you again take a lone of \$1000 with an annual interest rate of 5% but this time it is compounded annually. After the first year you will owe $\$1000 + \$1000 \cdot 0.05 = 1050$. After the second year you will owe $\$1050 + \$1050 \cdot 0.05 = 1,102.50$.

You may recognize this as a geometric series, the amount you owe at the end of a period is the amount that you owed at the end of the previous period multiplied by a common ratio

$$A_t = A_{t-1}(1 + r) \tag{8.10}$$

Thus we write the equation in general form

$$A_t = P(1 + r)^t \tag{8.11}$$

Again this does not tell the entire story because the interest rate is almost always reported as an annual figure and time is always given in years. Thus if R is the annual interest rate and there are n periods in a year we have the equation

$$A_t = P\left(1 + \frac{R}{n}\right)^{nt} \tag{8.12}$$

Notice now that the adjustments do not cancel out like they did in the case of simple interest. We can actually see that as loan is compounded more frequently, the value of the loan increases faster.

8.3.1 Example 1

Suppose you take out a car loan for \$25,000 which is compounded monthly with a interest rate of 4%. If you don't make any payments on the loan for 6 years, how much will the value of the loan be at that time?

$$A_6 = \$25,000\left(1 + \frac{0.04}{12}\right)^{12\cdot 6} = \$29,525 \tag{8.13}$$

Thus after 6 years, you will owe \$4525 more than you did when you took out the loan.

8.3.2 Example 2

As before, an investment and a loan are the same thing just from different perspectives, If you put \$8,000 in a certificate of deposit (a special kind of high interest bank account) for 5 years with an interest rate of 6% compounded quarterly, how much will you have when it comes time to withdraw the funds?

$$A_5 = \$8000\left(1 + \frac{0.06}{4}\right)^{5\cdot 4} = \$10,774 \tag{8.14}$$

8.3.3 Example 3

The difference right now between simple and compound interest may not be clear so lets do a comparison. Suppose that You invest \$3000 into two different investments, The first has a simple interest rate of 4% and a period of 1 month, the second also has a rate of 4% but it is compounded monthly. Lets compare the value of each investment after 40 years

$$\begin{aligned} A_4^s 0 &= \$3000(1 + 0.04\cdot 40) = \$7,800 \\ A_4^c 0 &= \$3000\left(1 + \frac{0.04}{12}\right)^{12\cdot 40} = \$14,819 \end{aligned} \tag{8.15}$$

See that we have the compound investment is worth almost double the value of simple investment. This is great when you are the one earning the interest, it isn't so nice when you owe interest.

8.4 Annuity

We so far have only discussed investments wherein you make the initial investment and leave it alone until a certain time when you take your money out and are left with whatever the value of the investment was at that time. Now we want to consider what happens if you continue to pay into an investment account regularly. This idea is what we call an annuity. An **Annuity** is a type of investment wherein you pay a regular amount at the *end* of a period for some amount of time.

8.4.1 Example 1

Suppose you are saving in a retirement account with 2% compound interest. You are going to put in \$5000 into this account every year for the next 10 years. At the end of these 10 years, what will the value of the account be? In order to solve this problem we will split our annuity into 10 separate compound interest investments.

The first investment of \$5,000 stays in the account and gains interest for 9 years (because it was added at the end of the first period) so from our findings in the previous chapter we know that the value of that investment is

$$A_9^{(1)} = 5,000(1 + 0.02)^9 = 5,000\cdot 1.02^9 \tag{8.16}$$

The first investment of \$5,000 stays in the account for only 9 years so the value of the investment is

$$A_8^{(2)} = 5,000(1+0.02)^8 = 5,000 \cdot 1.02^8 \tag{8.17}$$

We can continue with these computations to find the values of each of the \$5,000 installments after 10 years of investing like this

$$\begin{aligned} &A_9 = 5,000 \cdot 1.02^9 \quad A_8 = 5,000 \cdot 1.02^8 \quad A_7 = 5,000 \cdot 1.02^7 \\ &A_6 = 5,000 \cdot 1.02^6 \quad ... \quad A_1 = 5,000 \cdot 1.02 \quad A_0 = 5,000 \end{aligned} \tag{8.18}$$

The total value of the ordinary annuity is the sum of all of these investments separately

$$F = 5000(1.02^9 + 1.02^8 + ... + 1.02 + 1) \tag{8.19}$$

We are so close to an equation but in order to finish the job we need to take a break to talk about something called the geometric series. The **geometric series** is what happens when you add together all the terms of a geometric sequence. We have a nice way to find the value of this sum

8.4.2 Lemma

The Geometric series

$$1 + r + r^2 + r^3 + ... + r^{N-1} = \frac{r^N - 1}{r - 1} \tag{8.20}$$

8.4.3 Proof

$$\begin{aligned} S &= 1 + r = r^2 + r^3 + .. + r^{N-1} \\ rS &= r + r^2 + r^3 + r^4 + ... + r^N \\ rS - S &= r + r^2 + r^3 + ... + r^{N-1} + r^N \\ &\quad - 1 - r - r^2 - r^3 - ... - r^{N-1} \\ S(r-1) &= r^N - 1 \\ S &= \frac{r^N - 1}{r - 1} \end{aligned} \tag{8.21}$$

Example 1 Continued

With this formula now we can an equation for the value of our annuity. Notice that

$$1.02^9 + 1.02^8 + ... + 1.02 + 1 = \frac{1.02^{10} - 1}{1.02 - 1} \tag{8.22}$$

So in total our ordinary annuity is worth

$$F_{10} = 5000 \frac{1.02^{10} - 1}{0.02} \tag{8.23}$$

Let us repeat this Process in a more general setting. Suppose you have an interest rate r with n periods in a year. You are making a payment of p into this account at the end of each period. Lets find how much the annuity will be worth in t years. The first installment of p at the end of the first period will be worth

$$A_{n\cdot t-1} = p\left(1+\frac{r}{n}\right)^{n\cdot t-1} \tag{8.24}$$

after t years. Likewise, the next installment will be worth

$$A_{n\cdot t-2} = p\left(1+\frac{r}{n}\right)^{n\cdot t-2} \tag{8.25}$$

If we continue this process for each of the $n \cdot t$ periods we can add them all together to find that

$$F = p\left(\left(1+\frac{r}{n}\right)^{n\cdot t-1} + \left(1+\frac{r}{n}\right)^{n\cdot t-2} + ... + \left(1+\frac{r}{n}\right) + 1\right) \tag{8.26}$$

No we use the fact about geometric series that we talked about to write our final equation

$$F_t = p\frac{\left(1+\frac{r}{n}\right)^{n\cdot t} - 1}{\frac{r}{n}} \tag{8.27}$$

This equation tells us the "future value" of the annuity

8.4.4 Example 2

Consider an ordinary annuity with a 2% interest rate and a quarterly period. Suppose you make a \$300 investment into this account every quarter, How much will it be worth in 10 years?

$$F_{10} = 300\frac{(1+0.02/4)^{40} - 1}{0.02/4} = 13247.65 \tag{8.28}$$

8.4.5 Example 3

For our next example, consider again a retirement account. You hope you have that account be work \$500,000 in 50 years. The account has an interest rate of 3% with a quarterly period. How much should you invest each quarter for the next 50 years in order to ensure the account is worth \$500,000 at the end of that time?

$$500000 = p\frac{(1+0.03/4)^{4\cdot 50} - 1}{0.03/4} = p \cdot 460.89 \tag{8.29}$$

Solving for p in the above equation gives the answer \$1,084.86. This is the amount that should be invested every quarter in order to reach the goal. Whereas the previous examples of this type have given unreasonable amounts to invest. This amount, while still quite high, is not entirely unreasonable.

There is another important concept relating to annuity which is called the present value. We will not discuss it in detail but I want you to be aware of it in case you are looking for resources online and get mixed up between present and future value. What we have discussed in this class is future value.

8.5 Economic decision making

Economic decision making is the heading I will give to this section to convince us that our discussion of game theory fits in financial math. Although it is not a perfect fit, "Economic Decision Making and the Theory of Games" is the most important work in game theory so here is as appropriate a place to discuss it as any.

Game theory is a collection of methods to investigate how and why rational players of games make decisions. First we need to introduce the idea of a game. A **game** consists of two or more players which may play some number of strategies to get some payoff in return.

8.5.1 Example 1

As an example consider the "matching pennies" game. The two player game is quite simple. Each player places a penny on the table hidden from the other player, they may put it heads up or tails up. If both players' pennies match then they both win \$1 but if they do not match they both lose -\$1.

The most convenient way to represent this information is as a table

	H	T
H	(1,1)	(-1,-1)
T	(-1,-1)	(1,1)

Figure 8.1: Alice's strategies are listed on side and Bob's strategies are listed on the top. In the chart are shown the payoffs for both players, (Alice first, then Bob).

In this example we can introduce the ideas of a **Best response**. A best response for a player is the strategy that would maximize their payoff if every other player played the same strategy again. So for instance, if Bob is playing T then Alice's best response is to play T. This brings us to the concept of **Nash Equilibrium**. A Nash Equilibrium (NE) is a state wherein neither player would benefit by changing their strategy. For instance in the matching pennies game (H, H) is a Nash equilibrium because if Bob changes his strategy he will have a lower payoff and if Alice changes her strategy then she too will have a lower payoff. In game theory a player is making decisions only out of self interest,

8.5.2 Example 2

The natural question is to ask if every game has a Nash equilibrium? The answer is they do but only kind of. Consider the so called "Penalty Kick game" In a penalty kick the kicker may choose to kick to the left of the keeper or to the right. The keeper may choose to jump left or jump right. If the keeper jumps to the side the kicker kicked to she wins but if she jumps to the opposite side, she looses. We can draw our payoff matrix like this What we notice is that

	L	R
L	(1,-1)	(-1,1)
R	(-1,1)	(1,-1)

Figure 8.2: In this game the keepers strategies are listed on the left and the kickers strategies or listed on the top. The payoffs are shown in the matrix (Keeper's first, then Kicker's)

there are no "pure" Nash equilibria in this game. If both players are choosing L then the kicker wants to change their strategy. If the kicker kicks L then the keeper wants to change their strategy to get to save L. Then the Kicker will kick R so the keeper will change to save R. There is no pure strategy on this payoff matrix where both players wouldn't want to change their strategy.

In a case like this a Nash equilibrium still exists but it is what we call a "Mixed strategy" nash equilibrium. It is not worth our time to learn how to calculate these so mixed strategy equilibrium but we can know with certainty that one exists because of Nash's theorem. **Nash's Theorem** states that in any two player non cooperative game there exists an strategy (perhaps a mixed strategy) which is a Nash equilibrium.

Sometimes these Nash Equilibria are not the **Socially optimal** strategy. The Socially optimal strategy maximizes the total payout for all the players involved. It is not always true that the Nash equilibrium is socially optimal. When this happens we call it a Dilemma. The most famous is the prisoner's Dilemma.

8.5.3 Example 3

The setting is this: There are two suspects accused of a crime. There is enough evidence to convict each of them with a small charge but there is not enough evidence to convict the larger more serious felony charge without a confession. The DA offers both suspects a deal. "If you confess that the other suspect was directly involved then you will be given immunity, if you don't talk you will be charged with the smaller crime and if your partner takes the same deal then you will be charged with the larger crime as will".

Each player has the same two options: to talk or to remain silent. If both remain silent then they will both only be charged with the minor offense and spend 3 year in prison. If only one talks the one who talked will not get any jail time but the one who was silent will spend 10 years in Jail. If they both talk then, although they were both granted immunity from the small charge, they will be charged with the major offense and spend 7 years in prison. Lets look at the payoff matrix:

	Talk	Silent
Talk	(-7,-7)	(0,-10)
Silent	(-10,0)	(-3,-3)

Figure 8.3: The payoff matrix with prisoner 1's strategies on the side and prisoner 2's strategies on the top. In the matrix we see the payoffs (Player 1's, then Player 2's)

In the payoff matrix we notice that the Socially optimal case would be for neither prisoner to talk. In this case the total payoff is -6. Unfortunately for the prisoners, that is not a Nash equilibrium. If prisoner 2 will not talk, then prisoner 1 would be better off talking so prisoner's 1 best response is to talk. If prisoner 1 talks then prisoner 2's best response is also to talk. Notice that both prisoners talking is a Nash equilibrium because neither prisoner would benefit by changing their strategy.

This is the dilemma, the social optimal would be both remaining silent but if each prisoner is left to their own devices and plays with self interest they will not do the socially optimal thing.

8.5.4 Relation to Economics

If we return to financial math, we have discussed topic which fall under the heading "micro economics" which focuses on the individual. "Macro economics" focuses on groups of individuals. For the most part, every decision that a person makes is made in their own best interest. In that sense every individual is playing in their own best response. Sometimes when everyone plays their best response things work well. This is the case when the social optimum is the Nash equilibrium.

8.5.5 Example 4

An example of this might be trade. If Alice is Produces bread, she may be willing to trade her product or be unwilling to trade her product. Likewise if Bob picks blueberries, he may be willing to trade them or he may be unwilling to trade them. If Alice does not trade away some of her bread much of it will go to waste. Likewise if Bob does not trade away his blueberries they will go bad. We can imagine the following payoff matrix

If they do not both agree to trade then they will get only the benefit from their own production but when they do agree to trade they get more benefit. The state wherein the both trade is a Nash equilibrium because neither could

	Trade	Don't trade
Trade	(10,10)	(5,3)
Don't trade	(5,3)	(5,3)

Figure 8.4: The payoff matrix with Alice's strategies on the side and Bob's strategies on the top

improve their payoff by changing their strategy. This Nash equilibrium is the social optimum because it maximizes the total payoff.

8.5.6 Example 5

The previous example is the ideal case but it is not always true. Take a resource extraction example. If two people can fish in a lake at a high rate or at a low rate we might run into a problem. If both players are fishing at a high rate then the lake will be "over fished" and the supply of fish will decrease but in any other case the fish will be available. We can put this in a payoff matrix like this

	High	Low
High	(4,4)	(7,3)
Low	(7,3)	(5,5)

Figure 8.5: A payoff matrix for the simplified "Tragedy of the commons" game

This game is called the tragedy of the commons because everyone working in their best interest without cooperating leads to a decrease in overall production.

Theses are some examples of how game theory can be used to identify helpful and unhelpful economies. No system can eliminate dilemmas in total so it is important to understand when self interest works for the common good and when it works against the common good.

Chapter 9

Logic

9.1 Introduction

Math, at its heart, is a system of logic. WE have occasionally in this class escaped the world of numbers an equations to talk about mathematical objects like graphs and games. These things are just as mathematical as the algebra or calculus which we typically portray as "math". Both of these are connected by their reliance on logic

You have been using logic since you began to think. Humans are quite good at understanding cause and effect. Our goal here will be able to write down logical statements, evaluate logical statements and identify where we use these systems of logic in the mathematics we've been discussing this year.

We start by defining what at **Statement** is. A Statement is an assertion that is either true of false. Some examples are "I live in Knoxville, Tennessee" or "I have never failed a math exam" Some statements like "I live in Knoxville" can be proven true. Some statements like "I have never failed a math exam" can be proven false by counterexample (Remember when we discussed counter examples in chapter 3).

In math when we want to represent a statement we use a capital letter from the beginning to the alphabet. I can say "Consider the statements A and B. so A and B are both statements which means they can either be true or false. With this notation we can introduce the idea of **Logical Operators**. Its an intimidating name but what we mean by logical operators is "AND" and "OR". We use these operators to combine statements.

9.1.1 Example 1

The Statements "I am wearing jeans" and "I am wearing a t shirt" are both proper statements on their own. The statement "I am wearing jeans AND I am wearing a tshirt" is also a statement which is either true or false. Likewise "I am wearing jeans OR I am wearing a tshirt is a statement which can be true of false.

9.1.2 Example 2

The only other operator we will talk about is the NOT operator. Again the statement "I am wearing jeans" can be true or false and the statement "I am NOT wearing jeans" is also a statement which can be true of false.

9.2 Truth tables

A	B	A AND B
T	T	T
T	F	F
F	T	F
F	F	F

Figure 9.1: the truth table for the operator AND

These operators can be confusing so we will introduce a tool to investigate them: the **Truth Table** is a way of organizing statements. Consider two statements A and B. These can each either be TRUE or FALSE. Is the statement A AND B TRUE or FALSE? Lets investigate with the following truth table.

In the truth table above we see the information contained in the "A AND B" statement written out for us. Below is the truth table for the other two operators

A	B	A OR B
T	T	T
T	F	T
F	T	T
F	F	F

Figure 9.2: the truth table for the operator OR

A	NOT A
T	F
F	T

Figure 9.3: the truth table for the operator NOT

These truth tables show how these operators can be used to make longer statements. We can also use truth table so analyze longer more complicated statements.

9.2.1 Example 1

Consider the statement (NOT A) AND (NOT B). We can make the a truth table wherein each column represents a step in determining when this statement is true and when it is false.

A	B	NOT A	NOT B	(NOT A) AND (NOT B)
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Figure 9.4: the truth table for the statement (NOT A) AND (NOT B)

We can start by considering a small part of the statement like NOT A or NOT B then we can combine those smaller pieces to get (NOT A) AND (NOT B).

9.2.2 Example 2

As another example, consider the statement NOT (A or B). We will do the same thing by first considering what A OR B is then using the NOT operation(Fig 41).

A	B	A OR B	NOT (A OR B)
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

Figure 9.5: the truth table for the statement (NOT A) AND (NOT B)

Not only is this another example of how we can use a truth table to build up a more complicated statement, We can also see that the results of NOT (A AND B) and (NOT A) AND (NOT B) are identical. No matter what A and B are the two compound statements will be the same. This is a demonstration of something called DeMorgan's Law. Knowing the statement of DeMorgan's law is not important, rather, knowing how we used a truth table to show that these two compound statements are the same is the important part here.

9.3 Conditional Statements

Mathematics is a system of logic but it would be uninteresting if all we did was discuss statements without considering if they are true or false. It happens often in math that we want to determine if a statement is true or false but we cannot say directly if that is the case. Instead we build to the truth little by little using **conditional statements**. A conditional statement is a statement in the form of A implies B. In symbols we write $A \Rightarrow B$ and in English we say "if A then B". You use these all the time; in face we use these as a class all the time. Think about solving a simple equation.

$$\begin{aligned} &2x + 4 = 8 \\ \Rightarrow &2x = 4 \\ \Rightarrow &x = 2. \end{aligned} \tag{9.1}$$

In each step of the problem you do an algebraic manipulation but what we are really saying is that the first line implies the second line which implies the last line. So another way to say this is "If $2x + 4 = 8$" then $x = 2$".

There is a lot that can be said about a conditional statement but let us start with the vocabulary. In the statement $A \Rightarrow B$, A is the **Hypothesis** and B is the **Consequence**. We say that a conditional statement "holds" or is true if, whenever the hypothesis is true, the consequence is also true. There are many ways to write a conditional statement so it is not always immediately obvious which part of the statement is the Hypothesis and which is the consequence.

9.3.1 Example 1

For instance in the statement "If it rains then I will wear a raincoat" the Hypothesis comes before the consequence. However, we can also write this as "I will wear a raincoat if it rains". In this case the consequence comes first then the hypothesis.

9.3.2 Examples 2-5

It is a helpful exercise to consider the following conditional statements to identify the hypothesis and the consequence:

- If I don't water my plant, it will die.
- I'll buy eggs if they are on sale.
- If I haven't already filled out my Instructor Evaluation, I will fill it out later today
- My neighbor's dog barks whenever I open my door.

It is not always obvious which is the hypothesis but we can identify it by thinking "which of these statements will lead to the outcome".

Lets talk about that statement

$$A \Rightarrow B$$

If A is TRUE then B must be TRUE. Crucially this does not mean that if A is FALSE then B is FALSE. We need a stronger statement in order to say that. If the Hypothesis is not true we don't know anything about the consequence. It could be TRUE or FALSE.

9.4 Converse

For the conditional statement $A \Rightarrow B$. We define the **Converse** as the related statement $B \Rightarrow A$. If the original conditional statements holds, it means nothing for the converse. In other words: If A implies B, B does not necessarily imply A. We can see this through several examples.

9.4.1 Example 1

The first is the English sentence "If it is a weekend, I do not come into the office." The Hypothesis is "it is a weekend" and the consequence is "I do not come into the office." The converse of this statement is "If I do not come into the office then it is a weekend." Now the hypothesis and the consequence are switched. This new conditional statement does not hold. a counter example would be Thanksgiving. I do not come into the office on the 4th Thursday in November but it is not a weekend.

It would be a good exercise to practice forming the converse with the four statements used as examples above.

9.4.2 Example 2

Another more mathematical example would be $x = 2 \Rightarrow x^2 = 4$. This conditional statement holds but its converse is not necessarily true. The converse of this statement is $x^2 = 4 \Rightarrow x = 2$. This is not the case because if $x = -2$ then $x^2 = 4$. So the hypothesis can be satisfied without the consequence being true.

9.5 Contrapositive

For the conditional statement $A \Rightarrow B$ we define the **Contrapositive** as the statement NOT $B \Rightarrow$ NOT A. Unlike the converse, when the original statement holds, the contrapositive also holds. If A implies B, this means that whenever A is true then B must be true. There is no way for A to be true and for B to be False so if B is false then A must also be false. Therefore, when A implies B, if B is false then A is false.

Note that this is easily confused for the statements $B \Rightarrow A$ or NOT $A \Rightarrow$ NOT B. The contrapositive is crucially different than either of these statements.

9.5.1 Examples 1-3

Let us take a moment to figure out the contrapositive to each of the following conditional statements:

- If I don't water my plant, it will die.
- I'll buy eggs if they are on sale.
- My neighbor's dog barks whenever I open my door.

The answers are:

- If my plant did not die, I watered it.
- If I did not buy eggs, then they were not on sale
- If my neighbors dog does not bark, I did not open my door.

9.5.2 Example 4

My favorite exploration of contrapositive comes in the form a deceptively simple set up: Imagine I have four cards on a table. One is face up with a king on the front. One is face up with a 2 on the front. One is face down with a blue back, and the last is face down with a red back. If I tell you: Every card with a red back has a king on the front, which cards do you need to flip over in order to determine if this conditional statement holds? Pause to think about this problem for a couple minutes.

The correct answer is that you need to flip the Red card and you need to flip the 2. Why is that? It is obvious that you need to flip the red card over. If you are considering the statement ”Red cards have kings on them” then we clearly need to confirm that the red card has a king on it. But why do you need to flip the 2? why not the King? We are only making the statement that Red implies King. We are not saying that King implies anything. All red cards are kings but some blue cards can also be kings. For this reason, flipping the king tells us nothing.

Flipping the two, however, gives us information about the contrapositive. If I say that the conditional statement ”Red implies King” holds, then also the contrapositive Not King implies Not Red (2 implies blue) must also hold. Flipping the 2 and seeing that the back is blue tells us that the contrapositive holds.

In general when you seek to prove something you can either prove if that the original conditional statement is true or that the contrapositive is true. You need not prove both because if one holds then the other holds. If we are just investigating the situation case by case, in each case we can check the direction Red implies King or Not King implies Not Red. We can also say that we do not need to check a particular case because it does not matter in the conditional statement.

This is an interesting investigation of the contrapositive, it’s beyond the level that I would ask on an exam but it is interesting because it challenges our way of thinking.