Replicator Dynamics for Games on Networks

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Coordination Game

A coordination game is any game wherein using the same strategy as your coplayers improves your payoff. Characterized by the Bandwagon property[1]

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The most simple example is a normal form game where the payoff matrix is the identity matrix.

[1]Kandori, M. and Rob, R. (1998) Bandwagon effects and long run technology choice *Games and Economic Behavior*

Coordination Game

The consensus equilibrium is easy to understand but non-consensus equilibria are hard to identify

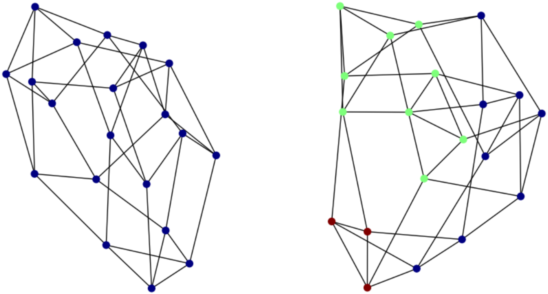


Figure: Taken from McAlister, J.S. and Fefferman N.H (2025) Insights into the structured coordination game with neutral options through simulation. *Dyamic Games and Applications*

Dynamic Game

Beginning in the 1990s with Kandori Milath and Rob(1993), Ellison (1993), Robson and Vaga-Redando (1995) **myopic best response with sequential update** was used, but they ran into challenges:

- The Best Response operation is highly discontinuous and set valued.
- The same analysis does not hold true for simultaneous update.

More recently, Swenson et al. (2018) So and Ma (2025), Ashkenazi-Golan et al. (2025) have made progress in circumventing these issues, but they still persist.

Evolutionary Games

Our approach to circumvent these issues is to adapt the well understood replicator equation to represent a structured population of individuals each using mixed strategies in a linear normal form game.

Replicator Equation

In the standard setting, the replicator equation looks like

$$\frac{d}{dt}p_i = p_i(f_i(p) - \varphi(p))$$

Where p is the mixture of strategies in the entire population, p_i is the proportion of the population playing strategy i, $f_i(p)$ is the fitness of playing strategy i and $\varphi(p)$ is the average fitness in the population.

The Model

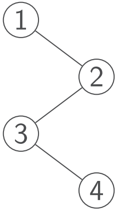
Consider an additive normal form game with payoff matrix A played on a static network with a set of players V with an adjacency matrix W. Each player has a mixed strategy u_v wherein u_v^i is the probability of playing strategy i.

Structured Replicator Equation

$$\frac{d}{dt}u_v^i = u_v^i \left\langle \hat{e}^i - u_v, \underbrace{\sum_{w \in V} W_{v,w} A u_w}_{g_v} \right\rangle$$

This is qualitatively different from what we think of as structured replication through pairwise proportional imitation (e.g. Ohtsuki and Nowak (2006) or Lieberman et al. (2005))

Example



$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

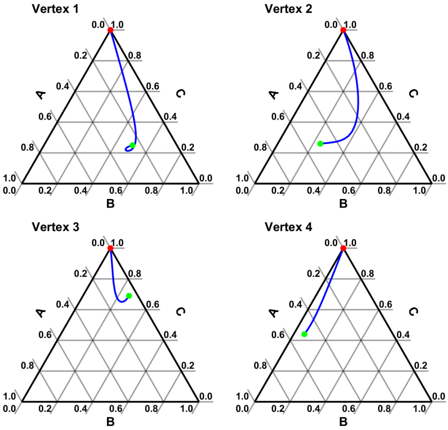
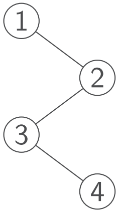


Figure: The coordination game played on P_4 converges to a consensus equilibrium from almost everywhere in the phase space.

Example



$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

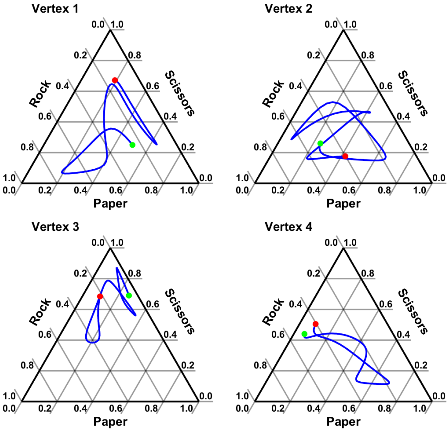


Figure: Rock Paper Scissors played on P_4 will not converge to an equilibrium.

Well Posedness

Lemma 1: Invariant Manifold

If $u(t)$ solves the IVP with $u_v(0) = \underline{u}_v \in \Delta^{m-1}$ for all v, then the solution exists for all time and $u_v(t) \in \Delta^{m-1}$.

Lemma 2: Better Reply Dynamic

If u solves the IVP with $u_v(0) = \underline{u}_v \in \Delta^{m-1}$ for all $v \in V$ then

$$\frac{d}{d\zeta} w_v(u_v + \zeta \frac{d}{dt} u_v | u) \geq 0$$

with equality only when $\frac{d}{dt} u_v = \vec{0}$.

Folk Theorem of Evolutionary Game Theory

Theorem 1: Folk Theorem of Evolutionary Game Theory

For the ODE model in the domain $(\Delta^{m-1})^V$, each of the following is satisfied

a. A stable rest point of the ODE system is a Nash equilibrium
b. A convergent trajectory of the ODE system converges to a Nash equilibrium
c. A strict Nash equilibrium is locally asymptotically stable

Is this model useful?

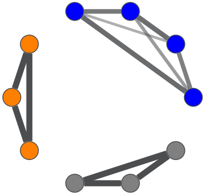
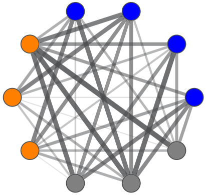
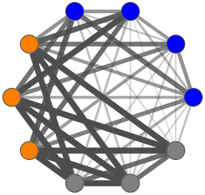
Use as an evolutionary model

It depends strongly on the time scale. Evolution is a population level process, so it may not always make sense to consider it on an individual level.

We can use this model to describe social learning with the assumption that ones ability to modulate the use of a strategy is proportional to their familiarity with the strategy. However, this model is useful in other contexts:

- Connection to the discrete time game
- Extendable to Evolutionary games in continuous player spaces

Optimally Stable Networks for Particular Strategy Profiles

A	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 10 & 1 & -10 \\ 1 & 0 & 1 \\ -10 & 1 & 10 \end{bmatrix}$
$W^\star$			
λ_0	-2	0	-5.46

Understanding Limit Cycles in Potential Games

Suppose that the game is a potential game with a potential function $\mathcal{W}$.

Lemma 3: Increasing potential

If $u(t)$ solves the IVP with $u(0) \in (\Delta^{m-1})^V$ then $\frac{d}{dt}\mathcal{W}(u(t)) \geq 0$ with equality only when $\frac{d}{dt}u(t) = 0$.

This means that for potential games, there are no limit cycles.

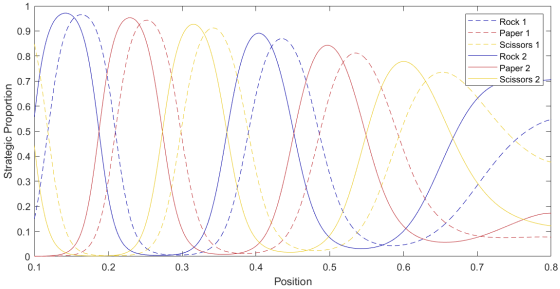
Conjecture: McAlister and Fefferman (2025)

If, under a simultaneous myopic best response dynamic, a potential game admits an orbit with a cycle. The period of that cycle is 1 or 2.

This was proven for two player games by Ashkenazi-Golan et al. (2025)

Extensions in Continuous Player Spaces

This model does become evolutionarily meaningful if we consider the limiting process from a graph to a continuous domain



$$g_v(u) = \sum_{w \in V} W_{v,w} A u_w \text{ becomes } g[u](x) = \int_\Omega K(x,y) A u(y) dy$$

Thank you, Any questions?

To see the work that I've done on this and other structured game theoretic projects, check out my website at jmcalis.github.io

Or email me at jmcalis6@vols.utk.edu