Replicator dynamics with spatial structure for evolutionary games

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Evolutionary Games

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Evolutionary game theory has many applications in all sorts of complex systems. It is important to the APPEX center because it comes up in

- Evolutionary Ecology
- Behavioral Ecology
- Collective Behavior
- Economics

Replicator Dynamics

The most famous way to describe an evolutionary game is through the replicator equation. If you have a well mixed infinitely large population and each player plays a single pure strategy then the proportion of players using the ith strategy, p_i changes in time by

$$\frac{d}{dt}p_i(t) = p_i(f_i(p) - \varphi(p))$$

where $p = [p_i]_{i=1}^m$, f_i is the fitness of playing strategy i against the mixture p and $\varphi(p)$ is the fitness of an average individual in the mixture p.

Replicator Dynamics

In the pure strategy coordination game your fitness is exactly the proportion of neighbors using the same strategy as you. Thus $f_i(p) = p_i$ and $\varphi(p) = |p|^2$

Coordination Example

For a pure coordination game. The replicator equation is

$$\frac{d}{dt} p_i = p_i(p_i - |p|^2)$$

All equilibria have the property that for all i such that $p_i \neq 0$ it is true that $p_i = |p|^2$. It is unstable when there is more than one such p_i.

Shifting Key Assumptions

Consider the setting where each player takes on a *mixed strategy*, there are a finite number of players where are not well mixed.

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$$\frac{d}{dt}u_v^i = u_v^i(f_v^i(u_v) - \varphi_v(u_v))$$

Is this model still meaningful?

Properties of the model

I argue that it is meaningful.

Well posedness

For an initial value problem $\frac{d}{dt}u_v^i = u_v^i(f_v^i(u_v) - \varphi_v(u_v))$ where $u_v^i(0) = u_{v0}^i$ and $u_{v_0} \in \Delta^{m-1}$ then $u_v(t) \in \Delta^{m-1}$ for all time $t > 0$

Better reply dynamic

At any time for a solution to the initial value problem above, all players are changing their strategy to increase their fitness relative to the present strategy profile. That is: $\langle \frac{\partial}{\partial t} u_v, \nabla w_v(u_v|u) \rangle \geq 0$ where $w_v(u_v|u)$ is the fitness of player v playing strategy u_v against the strategy profile u.

Computing the Jacobian

For a general normal form game with additive payoffs we write

$$\frac{d}{dt}u_v^i = f_v^i(u) = u_v^i(e^i - u_v)^\intercal A u_{\Gamma(v)} \tag{1}$$

where $u_{\Gamma(v)} = \sum_{w \in V} W_{wv} u_w$ For the pure coordination game in particular we can write

$$\frac{d}{dt}u_v^i = f_v^i(u) = u_v^i \langle e^i - u_v, u_{\Gamma(v)} \rangle \tag{2}$$

so we have $m \times n$ equations. We will order them first by strategy then by player so we write

$$\frac{d}{dt}u = [f_1^1, f_1^2, \ldots, f_1^m, f_2^1, f_2^2, \ldots f_2^m, \ldots, f_n^1, \ldots, f_n^m]^\intercal$$

Analyzing the Jacobian

For pure strategy equilibria we know that $u_v^i u_v^j = \delta_{i,j}$ and thus $(\nabla_w f_v^\intercal)^\intercal = 0$ whenever $w \neq v$ which is great because the Jacobian becomes block diagonal

$$J(u) = \begin{bmatrix} diag(\langle e^i - u_1, u_{\Gamma(1)} \rangle) - u_1 u_{\Gamma(1)}^\intercal & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & diag(\langle e^i - u_n, u_{\Gamma(n)} \rangle) - u_n u_{\Gamma(n)}^\intercal \end{bmatrix}$$

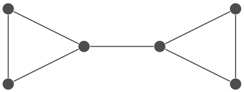
Analyzing the Jacobian

This is important because now we have a method to directly determine the stability of a coordinating system (indeed any matrix game) in linear time for 4 or fewer strategies.

1. We get local stability information easily
2. Perviously checking stability was a quadratic process
3. For more than 4 strategies, the eigenvalues cannot be computed in linear time but can we can find them in $\mathcal{O}(nm^2)$

Analyzing the Jacobian

As a sanity check we can see that the consensus equilibrium ($u_v^\star = [1, 0, ..., 0]^\intercal$ for all v) has $J(u^\star) = -I_n$ so it is clearly stable.



For the above graph we can easily compute that
$\hat{u} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$ is stable because $\sigma(J(\hat{u})) = \{-2, -1\}$

Analyzing the Jacobian

For the above graph we can easily compute that $\hat{u} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ and the spectrum is $\sigma(J(\hat{u})) = \{-2, -1, 0\}$.
The 0 eigenvalues correspond to non-asymptotic stability in this case. We do not yet know if that correspondence is general.

Continuous space extension

Instead of a discrete system, we can take a non-local extension, replacing the sum over an adjacency matrix with an integral over some familiarity kernel and get a nonlocal equation

$$\frac{\partial}{\partial t} u^i(x,t) = u^i(x,t)\langle e^i - u(x,t), AK * u(x,t)\rangle$$

We may even take the typical zero horizon limit and say $\Delta u^i \approx K * u^i - u^i$ to write the system as a PDE

$$\frac{\partial}{\partial t} u^i(x,t) = u^i(\langle e^i - u(x,t), A\Delta u(x,t)\rangle + \langle e^i - u(x,t), Au(x,t)\rangle)$$

Next steps

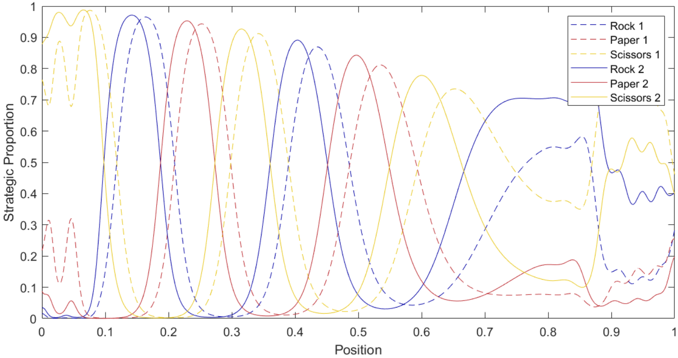


Figure: Continuous formulation admit interesting behavior like the traveling wave behavior shown here

Thank you

Questions?